Overview

Eigenvalues and eigenvectors appear in many problems. Here are just two that we will be interested in:

- What are the “fixed points” of a transformation — i.e. what points, lines or directions remain unchanged by the transformation?

- How stable is a solution to a minimization problem?

Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors are only defined for square matrices.

- The vector $x$ is an eigenvector of matrix $A$ and the scalar $\lambda$ is an eigenvalue of $A$ if

\[ Ax = \lambda x. \]

- We can show that equivalently, the scalar $\lambda$ is an eigenvalue of $A$ if

\[ \det(A - \lambda I) = 0. \]

(This is the characteristic equation for $A$.) and $x$ is an eigenvector of $x$ if $x$ is in the nullspace of $A - \lambda I$.

Example: Rotation

Consider the following rotation matrix:

\[
\begin{pmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

- This matrix has 1 real and 2 complex eigenvalues.

- The eigenvector corresponding to the real eigenvalue is $(0, 0, 1)^T$, which is the $z$-axis

- Points along this axis are unchanged by the rotation, and therefore these are “fixed points” of the rotation.

- All rotations in $\mathbb{R}^3$ have this same structure: a single fixed axis — the axis of rotation.
Properties of Eigenvalues and Eigenvectors

- Suppose λ₁, . . . , λₙ are the eigenvalues of A, then
  \[ \sum \lambda_i = \text{trace}(A), \quad \text{and} \quad \prod \lambda_i = \det(A). \]

- For a triangular matrix, the eigenvalues are the diagonal elements.

- A and A² have the same eigenvectors and if λᵢ is an eigenvalue of A, then λᵢ² is an eigenvalue of A².

- If A = M⁻¹BM for invertible matrix M, then A and B have the same eigenvalues, but not the same eigenvectors.

Computing Eigenvalues and Eigenvectors

- There is no simple form for computing eigenvalues and eigenvectors because there is no expression for the roots of a quintic or higher polynomial...

- Suppose n × n matrix A has a linearly independent set of eigenvectors x₁, . . . , xₙ, with associated eigenvalues λ₁, . . . , λₙ. Form the matrix
  \[ P = (x_1, x_2, \ldots, x_n). \]
  Then,
  \[ P^{-1}AP = \text{diag}(\lambda_1, \ldots, \lambda_n). \]
  Eigenvalue / eigenvector problems are generally solved by constructing such a matrix P for a given matrix A.

Symmetric Matrices and Spectral Decompositions

- The eigenvalues of symmetric matrices are all real, though not necessarily positive. This is easy to prove and we will do so in class.

- The eigenvectors corresponding to distinct eigenvalues of a symmetric matrix are orthogonal. You will have to prove this as a homework problem!

- Any symmetric n × n matrix A can be written as
  \[ A = VDV^T = \sum_{i=1}^{n} \lambda_i v_i v_i^T, \]
  where V is an orthogonal matrix whose columns are normalized eigenvectors vᵢ of A, and D is a diagonal matrix containing the corresponding eigenvalues λ₁, . . . , λₙ.
Quadratic Forms and Positive Definite Matrices

• For symmetric matrix \( A \), the expression \( x^T A x \) is called a quadratic form.

• Matrix \( A \) is said to be positive definite if
  \[ x^T A x > 0 \quad \text{for all non-zero} \quad x, \]
  or, equivalently, all eigenvalues of \( A \) are positive.

• If all eigenvalues are non-negative or, equivalently, \( x^T A x \geq 0 \), then \( A \) is positive semi-definite.

• Any matrix of the form \( A = B^T B \) is positive semi-definite.

• When the Hessian matrix of a function minimization problem is positive definite, the solution is stable.
Practice Problems / Potential Test Questions

1. Suppose that $A$ and $B$ are square matrices and $v$ is an eigenvector of $A$. We know that $v$ is also an eigenvector of $A^2$. Is $v$ also an eigenvector of $C = BA$? Prove your answer.

2. Prove that the eigenvectors of distinct eigenvalues of a $2 \times 2$ matrix are linearly independent.

3. Show that
   \[ x^T A x > 0 \]
   for symmetric matrix $A$ and all vectors $x$ implies that all eigenvalues of $A$ are positive.

Problems For Grading

Submit solutions to the following problems on Monday, February 13th, as the first part of HW 4.

1. (10 points) Prove that the eigenvectors of distinct eigenvalues of a symmetric matrix are orthogonal to each other.

2. (10 points) How can you find the nullspace of a square matrix from its eigenvalues and eigenvectors?