Overview of Today’s Lecture

- Shortened discussion from Lecture 9 of PCA and its application to recognition and shape modeling.
- Introduction to projective geometry in 2D.
  - This material is covered in Chapter 1 of Hartley and Zisserman, although we are motivating it differently here.
  - The emphasis is homogeneous representations.
  - Lecture 11 will cover projective transformations.

Vanishing Points

- A simplified formulation of the perspective (pinhole) projection of a point \((x, y, z)^\top\) is
  \[
  (u, v)^\top = \left( \frac{fx}{z} \quad \frac{fy}{z} \right)^\top, \tag{1}
  \]
  where \(f\) is the focal length of the camera. This can be demonstrated using similar triangles assuming that all projection line pass through the origin and intersect the camera plane at \(z = -f\).

- Recall that the parametric equation of a line in the world is
  \[
  \mathbf{x}(t) = \mathbf{x}_0 + t \mathbf{d} \tag{2}
  \]
  where \(\mathbf{x}_0 = (x_0, y_0, z_0)^\top\) and \(\mathbf{d} = (d_x, d_y, d_z)^\top\).

- The parametric image equation for the projection of this line is
  \[
  (u(t), v(t)) = \left( \frac{f(x_0 + td_x)}{z_0 + td_z}, \frac{f(y_0 + td_y)}{z_0 + td_z} \right). \tag{3}
  \]
  As you are asked to prove in the homework, as \(t\) varies this really does form a line in 2D!

- Now consider what happens to the image point as \(t \to \infty\).
  - We get the image point
    \[
    \left( \frac{fd_x}{d_z}, \frac{fd_y}{d_z} \right) \tag{4}
    \]
The image projection of the limit is a real image point, called a “vanishing point,” despite the fact that the limiting point in the world is infinitely far from the camera.

• Notice that any other line parallel to this one has the exact same vanishing point.
• We need a geometry that is able to describe “points at infinity” — intersections of parallel lines — and horizon lines, which are “lines at infinity”.
• Projective geometry allows us to do this.
• We’ll start with two-dimensional projective geometry — the space \( \mathcal{P}^2 \).

**Lines and Points; Homogeneous Representations**

• As we have already discussed this semester, the line equation, 
  
  \[ ax + by + c = 0, \]
  
  can be written as the dot product:
  
  \[
  \begin{pmatrix}
  a & b & c
  \end{pmatrix}
  \begin{pmatrix}
  x \\
  y \\
  1
  \end{pmatrix}
  = 0 \tag{6}
  \]

• The vector of line parameters \( l = (a, b, c)^T \) is unique only up to a (non-zero) scale factor.
• We can also write the point as a vector \( x = (x, y, 1)^T \), and then write \( x = (\lambda x, \lambda y, \lambda)^T \), for \( \lambda \neq 0 \). This is the same point.
• More generally we write \( x = (x_1, x_2, x_3)^T \).
  
  – This is the *homogeneous* representation of the point.
  – We can recover the original \( x \) and \( y \) from the ratios \( x_1/x_3 \), \( x_2/x_3 \). The values \( x \) and \( y \) are the associated *affine coordinates* of the point.
• The line equations are now simply
  
  \[ l^T x = x^T l = 0. \] \tag{7}

**Intersections and Duality**

• For two distinct vectors of line parameters \( l_1, l_2 \), we can find the unique point on both lines from
  
  \[ x = l_1 \times l_2, \] \tag{8}
  
  or equivalently from the null space of
  
  \[
  L = \begin{pmatrix}
  l_1^T \\
  l_2^T
  \end{pmatrix}
  . \tag{9}
  \]
• We can do the exact same thing to find the vector describing the line joining two points.

• These properties arise from the homogeneous equation $1^T x = 0$ and from the fact that we have written $x$ in homogeneous form.

• **Duality:** The symmetry between points and lines in all of the above expressions is exact. Any theorem in $\mathcal{P}^2$ about lines is true about points and vice-versa.

### Points at Infinity; Line at Infinity

• The intersection point between $l_1 = (a, b, c)^T$ and $l_2 = (a, b, d)^T$ is $x = (-b, a, 0)^T$.

• Points with 0 as their last component are “points at infinity”. These points have no affine analog.

• There is a 1-1 correspondence between line directions and points at infinity in $\mathcal{P}^2$.

• The line $l_\infty = (0, 0, 1)^T$ is the “line at infinity”. All points at infinity are on this line.

• In projective geometry there is nothing special about points and lines at infinity. Only when we try to map them into an affine world do they become special (non-mappable).

### Conics

• You probably learned the following general equation of a conic

$$ax^2 + bxy + cx^2 + dx + ey + f = 0.$$  \hspace{1cm} (10)

• This can be written as a quadratic form involving a symmetric matrix:

$$
\begin{pmatrix}
x & y & 1
\end{pmatrix}
\begin{pmatrix}
a & b/2 & d/2 \\
b/2 & c & e/2 \\
d/2 & e/2 & f
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
1
\end{pmatrix} = 0,
$$

or more simply,

$$x^T C x,$$ \hspace{1cm} (12)

where $C$ is symmetric and rank 3.

• Five points determine a conic. We can find this conic (the parameter vector $(a, b, c, d, e, f)^T$) from the null space of

$$
\begin{pmatrix}
x_1^2 & x_1 y_1 & y_1^2 & x_1 & y_1 & 1 \\
x_2^2 & x_2 y_2 & y_2^2 & x_2 & y_2 & 1 \\
x_3^2 & x_3 y_3 & y_3^2 & x_3 & y_3 & 1 \\
x_4^2 & x_4 y_4 & y_4^2 & x_4 & y_4 & 1 \\
x_5^2 & x_5 y_5 & y_5^2 & x_5 & y_5 & 1
\end{pmatrix}.
$$ \hspace{1cm} (13)
• Although it is likely that the conics you have dealt with in the past create rank 3 conic matrices, conic matrices of rank less than 3 are sometimes of interest.

• The line \( \mathbf{l} = \mathbf{C}\mathbf{x} \) is tangent to the conic at point \( \mathbf{x} \).

• There is the notion of a “dual” or line conic for a given (point) conic. This conic is \( \mathbf{C}^* \) — the adjoint of \( \mathbf{C} \). If \( \mathbf{C} \) is rank 3, \( \mathbf{C}^* \) equals \( \mathbf{C}^{-1} \), up to an irrelevant scale factor.

• Line \( \mathbf{l} \) is tangent to \( \mathbf{C} \) if and only if
  \[
  \mathbf{l}^\top\mathbf{C}^*\mathbf{l} = 0.
  \] (14)

### Homogeneous Forms and Scale Factors

• One of the hardest things to get used to in projective geometry is that representations are no longer unique. In particular:
  
  – Two points, \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \), are equal if \( \mathbf{x}_1 = \lambda \mathbf{x}_2 \) for real number \( \lambda \neq 0 \).
  
  – Two lines, \( \mathbf{l}_1 \) and \( \mathbf{l}_2 \), are equal if \( \mathbf{l}_1 = \lambda \mathbf{l}_2 \) for real number \( \lambda \neq 0 \).
  
  – Two conics, \( \mathbf{C}_1 \) and \( \mathbf{C}_2 \), are equal if \( \mathbf{C}_1 = \lambda \mathbf{C}_2 \) for real number \( \lambda \neq 0 \).

• These arise from our desire to represent points and lines at infinity and because all equations have a homogeneous form:
  
  \[
  \mathbf{l}^\top\mathbf{x} = 0 \quad \text{and} \quad \mathbf{x}^\top\mathbf{C}\mathbf{x} = 0.
  \]

  – One implication of this is that a conic matrix only has 5 degrees of freedom.

• Counting degrees of freedom in projective geometry and in other problems can be tricky. It is always useful however as (a) a sanity check, and (b) to verify your understanding.
Practice Problems and Potential Test Questions

1. Find the intersection point of lines $l_1 = (1, 0, 5)^T$ and $l_2 = (-3, 2, 4)$. What are the affine coordinates of this point?

2. Show that the null space of a $2 \times 3$ matrix is equal to the cross-product of the two row vectors.

3. At first, parametric equations of lines in $\mathcal{P}$ “feel funny”. But, parametric equations can be written in the usual way as a linear combinations of two points. In particular, if $x$ and $y$ are points on the line and $d$ is the (two component) tangent direction, then the line can be written as any one of the following:

$$p(\alpha) = x + \alpha y$$
$$p(\beta) = \beta x + (1 - \beta)y$$
$$p(\gamma) = x + \gamma \begin{pmatrix} d \\ 0 \end{pmatrix}$$

Prove that these forms are all equivalent. To do this, recall the conditions under which two points $x_1$ and $x_2$ are equal and recall the relationship between points at infinity and line directions.