Represent Links in TR^2

\[ T_2 = \begin{bmatrix} \hat{x}_2 & \hat{y}_2 & p_{200}^1 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ p^1 = T_2 \cdot p^2 \]

\[ T_2' = \begin{bmatrix} c_{\theta_1} & -s_{\theta_1} & d_{\text{ix}} \\ s_{\theta_1} & c_{\theta_1} & d_{\text{iy}} \\ 0 & 0 & 1 \end{bmatrix} = T_2'(\theta_1, d_{\text{ix}}, d_{\text{iy}}) \]

To implement revolute joint:

fix \( d_{\text{ix}}, d_{\text{iy}} \) and vary \( \theta_1 \)

To implement prismatic joint:

fix \( \theta_1 \) and vary \( d_{\text{ix}}, d_{\text{iy}} \) to lie along a line

\( d_{\text{iy}} = 3d_{\text{ix}} + 7 \)
Don't think of quaternion as operating on a rigid body rather as operating on a point.

If there is a point, then you specify $(\mathbf{n}, \theta)$

If you specify $(-\mathbf{n}, -\theta)$

then $\mathbf{p}$ moves to $\mathbf{p}'$

also.

Meet w/ Post to start project. Don't leave off until end.

Forgot to mention:

(1) $C$-space in general has $\#$ of components that is exponential in the dimension of $C$.

(2) Constructing all of $\text{Cobs}$ is impractical even in $\text{SE}(3)$.

(3) Number of features of $\text{Cobs}$ is polynomial in the $\#$ of verts, edge, faces of $A \times \mathcal{O}$. 3/25/08
Chapter 5: Sampling-Based MP

Avoid explicit construction of $C_{obs}$
Even with polyhedra
  $\#$ of components of $C_{free}$ in general is exp  
  the dimension of $C$.

$\#$ of facets of $C_{obs} = \binom{m+k}{k}$ where $k$ approx geom. complexity
  at least quadratic edges, faces, vertices

This approach assumes geometry determines goodness of plan.
Maybe energy use should be included.

Other things:

Completeness — Alg. is complete if it returns soln in finite time
  if one exists, or it returns non-existence in finite time.

Sampling-Based Methods are not complete but are more practical typically.
Resolution Complete

If samples cover C-space densely as $t \rightarrow \infty$, then Sample-Based methods are resolution complete.

Probabilistically Complete

If sampling is based on a probability dist. and if the sampling is dense over C, then it is prob complete.

Rate of convergence becomes important, but hard to establish!

5.1 Distance & Volume

We require a distance function on C for effective sampling.

C becomes a metric space

Means you can have a notion of distance.

Some measure of volume is also helpful $\Rightarrow$ C is a measure space

Means you can have a notion of volume.
Metric Spaces

$\mathbb{R}^n$ - the usual Euclidean norm

$$\|x_1 - x_2\| = \sqrt{(x_1 - y_1)^2 + (y_1 - y_2)^2 + \ldots}$$

On curved spaces, $\mathbb{C}$, we must have

Non neg. $\rho(a, b) \geq 0$ \hspace{1cm} a, b \in X

Reflexive $\rho(a, b) = 0$ only if $a = b$

Symmetric $\rho(a, b) = \rho(b, a)$

Triangle Ineq $\rho(a, b) + \rho(b, c) \geq \rho(a, c)$

$L_p$ Norms

$L_p$ Metrics on $\mathbb{R}^n$

$$\rho = \left( \sum_{i=1}^{n} |x_i - x'_i|^p \right)^{\frac{1}{p}} \quad , \quad p \geq 1$$

where $x, x' \in \mathbb{R}^n$

$L_2 = \text{usual Euclidean Metric}$

$L_1 = \text{Manhattan}$

$L_\infty = \max_{i=1}^{n} \{ |x_i - x'_i| \}$ limit as $p \to \infty$

Note: Metric Space is $(X, \rho) = \text{a topological space w/ a metric}$.

A product of metric spaces is a metric space.

$X = X_1 \times X_2 \times \ldots \times X_n$
Metric on \( \text{SO}(2) = \{(a,b) \in \mathbb{R}^2 \mid a^2 + b^2 = 1\} \)

L₂ metric: \( \rho(a_1,b_1; a_2,b_2) = \sqrt{(a_1-a_2)^2 + (b_1-b_2)^2} \)

Suppose we sample uniformly? What does that mean?

It means samples are equally distant, i.e., depends on metric chosen.

L₂ using \( a, b \) parametrization

Opposite points are maximally distant

2 points equally distant

\[ \rho(a_1, b_1; a_2, b_2) = |a_1 - a_2| + |b_1 - b_2| \]

\[ \rho(\theta_1, \theta_2) = \min \{ |\theta_1 - \theta_2|, 2\pi - |\theta_1 - \theta_2| \} \]

This assumes \( \theta_1, \theta_2 \in [0, 2\pi] \)

Really you need

\[ \rho(\theta_1, \theta_2) = \min \{ \mod(\theta_1 - \theta_2, 2\pi), \ldots, \mod(2\pi - \theta_1 + \theta_2, 2\pi) \} \]

\[ \rho(\theta, \theta) = \cos^{-1}(a.a + b.b) \]
Metric on $SE(2)$

Let $q \in \mathbb{R}^4$, $q = (x, y, a, b)$

$L_2$ metric on $\mathbb{R}^4$

$$p_{L_2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (a_1 - a_2)^2 + (b_1 - b_2)^2}$$

Alternative metric

$$p = c \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} + c_2 \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$$

Could use $L_2$ in $\mathbb{R}^2$ and $L_1$ in other $\mathbb{R}^2$

$L_p$ in $\mathbb{R}^2$ and $L_\infty$ in $S'$

Note: mismatch $L$ in $\mathbb{R}^2$ is in feet, meters, ...

$L$ in $S'$ is in radians, degrees, ...

Can be helpful to multiply $L$ in $S'$ by a characteristic distance or divide $L$ in $\mathbb{R}^2$ by a characteristic distance

This can be tricky to choose "right" characteristic length.

Imagine $S' \times \mathbb{R}'$

Suppose you grid $C$ for motion planning using constant distances between columns and rows.

Is this uniform?

Depends on metric.
Metric on $SO(3)$

Let $h_i$ be a unit quaternion

$$\rho(h_1, h_2) = \min \{ \| h_1 - h_2 \|, \| h_1 + h_2 \| \}$$

$takes care of wrap-around identification issue$

This metric has analogous problem as the metric on $SO(2)$ that cuts straight across the circle.

Fix problem with spherical linear interpolation (analogous to measuring distance along the great circle between two points on $S^3$ (embedded in $\mathbb{R}^4$)).

Let $p_5(h_1, h_2) = \cos^{-1}(a_1 a_2 + b_1 b_2 + c_1 c_2 + d_1 d_2)$

Then a good metric is:

$$\rho(h_1, h_2) = \min \{ p_5(h_1, h_2), p_5(h_1, -h_2) \}$$

The metric you choose should be an important quantity for the system in question.

**Ex. 5.6**

Robot Displacement (max displacement of point)

$$\rho(q_1, q_2) = \max_{a \in A} \{ a(q_1) - a(q_2) \}$$

\[ 1 \rightarrow \rho \rightarrow 1 \]
Ex 5.7 Metric on $T^n$

There are $n$ arbitrary coefficients

$$p = c_1 \sqrt{(q_1 - q'_1)^2} + c_2 \sqrt{(q_2 - q'_2)^2}$$

Maybe $c_i$ should be very small since $\Delta q_i \Rightarrow$ big motion at tip.

Should $c_1$ be small here?

What if robot is working close to base?

Seems that metrics may need to be configuration dependent if goal is to sample the workspace uniformly. But would we retain the metric property?

$$\Delta x = J \Delta q = J(q) \Delta q$$

Suppose we want to sample $X$ uniformly to...
Suppose we sample $q_1$

every $\frac{\pi}{2}$, $\left\{0, \frac{\pi}{2}, \pi, 3\frac{\pi}{2}\right\}$

Even if we sample $q_1$ finely, we will never find holes.

Metrics for $SE(3)$.

Same issue with units of $\mathbb{R}^3 \times SO(3)$

Can add any $SO(3)$ metric to an $\mathbb{R}^3$ metric.
C-space is a set with an uncountably infinite # of elements.

But sampling-based planning must terminate after a finite # of samples have been considered.

∴ Sampling techniques should be carefully designed.

Since samples are discrete, the planning/search alg of Ch 2 can be used.

Clearly performance is a function of sampling alg.

---

**Denseless**

A set \( U \) is dense in \( V \) if

\[ \text{cl}(U) = V \]

\[ e.g. \quad \begin{array}{c}
\quad U \\
\quad \quad V
\end{array} \quad (0, 1) \quad [0, 1] \]

Suppose \( V \) is C-space and \( U \) is set of samples. To guarantee completeness, samples must get arbitrarily close to every point in \( V \).

* Sampling methods must produce a dense set of samples.*
The van der Corput Sequence

Let $C = \mathbb{Q}_\mathbb{Z}$ be homeomorphic to $SO(2) = S^1$.

Suppose you want 4 evenly-spaced samples \{0, 1/4, 1/2, 3/4\}.

<table>
<thead>
<tr>
<th>Order of samples</th>
<th>Binary Rep</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00000</td>
<td>0</td>
</tr>
<tr>
<td>0.00010</td>
<td>1/16</td>
</tr>
<tr>
<td>0.00100</td>
<td>2/16</td>
</tr>
<tr>
<td>0.01000</td>
<td>1/4</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>0.11000</td>
<td>3/4</td>
</tr>
</tbody>
</table>

Suppose you want 16 samples.

van der Corput – flipped the bits

<table>
<thead>
<tr>
<th>Order of samples</th>
<th>Binary Rep</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00000</td>
<td>0</td>
</tr>
<tr>
<td>0.10000</td>
<td>1/2</td>
</tr>
<tr>
<td>0.01000</td>
<td>2/4</td>
</tr>
<tr>
<td>0.11000</td>
<td>3/4</td>
</tr>
<tr>
<td>0.00100</td>
<td>1/8</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

What really matters is order of samples.

If we take all samples, there are attempted connections.

What is high density?

What is low density?

Maintains sampling density in breadth-first manner!
van der Corput sequence benefits

- Need not determine # of samples in advance.
  Sequence is dense in $[0,1]/n$

Random Sampling.

Suppose we pick $k$ points at random $U[0,1]$. The probability that no points fall in an interval of length $e$ is $(1-e)^k$. This probability $\to 0$ as $k \to \infty$.

\[ \therefore \text{random sampling is dense (w/ probability 1).} \]
Random Sampling of $C$

Goal: $\text{Prob}(q)$ is constant over $C$.

Is that even good?

It may be the best that can be done w/o knowing more about the problem or the structure of $C$ (which we don't know in advance in detail).

Maybe local planner allows some error of line segment connections.

Local planner may disallow this.

Which is better?

Uniform sampling of $SO(3)$:

Uniform sampling of Euler angles does not give uniform sampling of orientations in $SO(3)$.

Uniform sampling of unit quaternions does!

Here's how. Choose $u_1, u_2, u_3 \in [0, 1]$ uniformly at random.

A uniform random quaternion is given by:

$$h = (\sqrt{1-u_1} \sin(2\pi u_2), \sqrt{1-u_1} \cos(2\pi u_2), \sqrt{u_1} \sin(2\pi u_3), \sqrt{u_1} \cos(2\pi u_3))$$

Imagine sampling uniformly at latitude and longitude. Sphere will be more densely sampled at the poles.

Eq. (5.15)
If \( \sin(2\pi \alpha_2) < 0 \), replace \( \sin(2\pi \alpha_2) \) with \( -\sin(2\pi \alpha_2) \).

Now, we can sample uniformly (deterministically or at random) over the interval, \( I, S', \text{SO}(3) \).

Since uniform random sample of \( X \times Y \) is just \( (x,y) \) where \( x \) is a uniform random sample of \( X \) and \( y \) is a uniform random sample of \( Y \).

\[ \therefore \text{We can sample uniformly of any } C \text{ composed as a product of intervals, } \text{circles, } \text{and } \text{SO}(3).\]

This is pretty much everything, but not all circles are equal.

If circle is a square, samples will be denser at the points closer to origin.

Question?

Is it better to choose \( k \) sample at random on \( I \) or is it better to choose them on the largest current subinterval?

\[ 0 \cdots 1 \]

\[ \cdots \text{What's the probability of the greatest distance being more than a given fraction of the interval?} \]