

Logic Programming (PLP 11)

Predicate Calculus, Horn Clauses,
Clocksin-Mellish Procedure

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An Early (1971) “Conversation”

USER:

Cats kill mice.

Tom is a cat who does not like mice who eat cheese.

Jerry is a mouse who eats cheese.

Max is not a mouse.

What does Tom do?

COMPUTER:

Tom does not like mice who eat cheese.

Tom kills mice.

USER:

Who is a cat?

COMPUTER:

Tom.

USER:

What does Jerry eat?

COMPUTER:

Cheese.

USER:

Who does not like mice who eat cheese?

COMPUTER:

Tom.

USER:

What does Tom eat?

COMPUTER:

What cats who do not like mice who eat cheese eat.

Another Conversation

USER:

Every psychiatrist is a person.

Every person he analyzes is sick.

Jacques is a psychiatrist in Marseille.

Is Jacques a person?

Where is Jacques?

Is Jacques sick?

COMPUTER:

Yes.

In Marseille.

I don't know.

Logic programming

- A program is a collection of *axioms*, from which theorems can be proven.
- A *goal* states the theorem to be proved.
- A logic programming language implementation attempts to satisfy the goal given the axioms and built-in inference mechanism.

Propositional Logic

- Assigning truth values to logical propositions.
- Formula syntax:

f	$::=$	v	symbol
		$f \wedge f$	and
		$f \vee f$	or
		$f \Leftrightarrow f$	if and only if
		$f \Rightarrow f$	implies
		$\neg f$	not

Truth Values

- To assign a truth values to a propositional formula, we have to assign truth values to each of its atoms (symbols).
- Formula semantics:

a	b	$a \wedge b$	$a \vee b$	$a \Leftrightarrow b$	$a \Rightarrow b$	$\neg a$
False	False	F	F	T	T	T
False	True	F	T	F	T	T
True	False	F	T	F	F	F
True	True	T	T	T	T	F

Tautologies

- A *tautology* is a formula, **true** for all possible assignments.

- For example: $\neg\neg p \Leftrightarrow p$

- The contrapositive law:

$$(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$$

- De Morgan's law:

$$\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$$

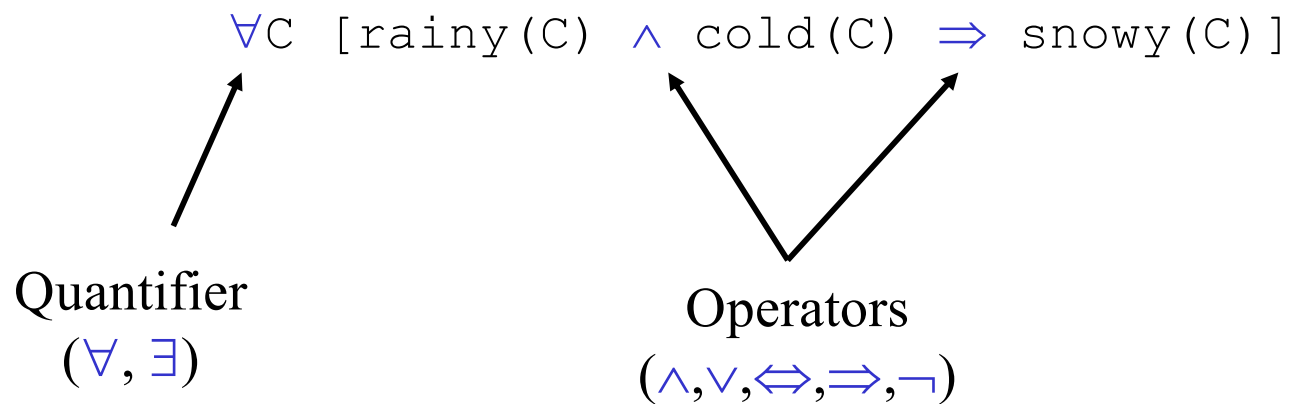
First Order Predicate Calculus

- Adds variables, terms, and (first-order) quantification of variables.
- Predicate syntax:

a	::=	$p(v_1, v_2, \dots, v_n)$	predicate
f	::=	a	atom
		$v = p(v_1, v_2, \dots, v_n)$	equality
		$v_1 = v_2$	
		$f \wedge f$ $f \vee f$ $f \Leftrightarrow f$ $f \Rightarrow f$ $\neg f$	
		$\forall v. f$	universal quantifier
		$\exists v. f$	existential quantifier

Predicate Calculus

- In mathematical logic, a *predicate* is a function that maps constants or variables to **true** and **false**.
- Predicate calculus enables reasoning about propositions.
- For example:



Quantifiers

- *Universal* (\forall) quantifier indicates that the proposition is true for **all** variable values.
- *Existential* (\exists) quantifier indicates that the proposition is true for **at least one** value of the variable.
- For example:

$$\forall A \forall B [(\exists C [\text{takes}(A, C) \wedge \text{takes}(B, C)]) \Rightarrow \text{classmates}(A, B)]$$

Structural Congruence Laws

$$P_1 \Rightarrow P_2 \equiv \neg P_1 \vee P_2$$

$$\neg \exists X [P(X)] \equiv \forall X [\neg P(X)]$$

$$\neg \forall X [P(X)] \equiv \exists X [\neg P(X)]$$

$$\neg (P_1 \wedge P_2) \equiv \neg P_1 \vee \neg P_2$$

$$\neg (P_1 \vee P_2) \equiv \neg P_1 \wedge \neg P_2$$

$$\neg \neg P \equiv P$$

$$(P_1 \Leftrightarrow P_2) \equiv (P_1 \Rightarrow P_2) \wedge (P_2 \Rightarrow P_1)$$

$$P_1 \vee (P_2 \wedge P_3) \equiv (P_1 \vee P_2) \wedge (P_1 \vee P_3)$$

$$P_1 \wedge (P_2 \vee P_3) \equiv (P_1 \wedge P_2) \vee (P_1 \wedge P_3)$$

$$P_1 \vee P_2 \equiv P_2 \vee P_1$$

Clausal Form

- Looking for a *minimal kernel* appropriate for theorem proving.
- Propositions are transformed into **normal form** by using structural congruence relationship.
- One popular normal form candidate is *clausal form*.
- Clocksin and Melish (1994) introduce a 5-step procedure to convert first-order logic propositions into clausal form.

Clocks in and Melish Procedure

1. Eliminate implication (\Rightarrow) and equivalence (\Leftrightarrow).
2. Move negation (\neg) inwards to individual terms.
3. *Skolemization*: eliminate existential quantifiers (\exists).
4. Move universal quantifiers (\forall) to top-level and make implicit, i.e., all variables are universally quantified.
5. Use distributive, associative and commutative rules of \vee , \wedge , and \neg , to move into *conjunctive normal form*, i.e., a conjunction of disjunctions (or *clauses*.)

Example

$$\forall A [\neg \text{student}(A) \Rightarrow (\neg \text{dorm_resident}(A) \wedge \neg \exists B [\text{takes}(A,B) \wedge \text{class}(B)])]$$

1. Eliminate implication (\Rightarrow) and equivalence (\Leftrightarrow).

$$\forall A [\text{student}(A) \vee (\neg \text{dorm_resident}(A) \wedge \neg \exists B [\text{takes}(A,B) \wedge \text{class}(B)])]$$

2. Move negation (\neg) inwards to individual terms.

$$\forall A [\text{student}(A) \vee (\neg \text{dorm_resident}(A) \wedge \forall B [\neg (\text{takes}(A,B) \wedge \text{class}(B))])]$$

$$\forall A [\text{student}(A) \vee (\neg \text{dorm_resident}(A) \wedge \forall B [\neg \text{takes}(A,B) \vee \neg \text{class}(B)])]$$

Example Continued

$$\forall A [\text{student}(A) \vee (\neg \text{dorm_resident}(A) \wedge \forall B [\neg \text{takes}(A, B) \vee \neg \text{class}(B)])]$$

3. *Skolemization*: eliminate existential quantifiers (\exists).
4. Move universal quantifiers (\forall) to top-level and make implicit, i.e., all variables are universally quantified.

$$\text{student}(A) \vee (\neg \text{dorm_resident}(A) \wedge (\neg \text{takes}(A, B) \vee \neg \text{class}(B)))$$

5. Use distributive, associative and commutative rules of \vee , \wedge , and \neg , to move into *conjunctive normal form*, i.e., a conjunction of disjunctions (or *clauses*.)

$$(\text{student}(A) \vee \neg \text{dorm_resident}(A)) \wedge (\text{student}(A) \vee \neg \text{takes}(A, B) \vee \neg \text{class}(B))$$

Horn clauses

- A standard form for writing axioms, e.g.:

`father(X, Y) \Leftarrow parent(X, Y), male(X) .`

- The Horn clause consists of:
 - A *head* or consequent term H , and
 - A *body* consisting of terms B_i

$H \Leftarrow B_0 , B_1 , \dots , B_n$

- The semantics is:

« If B_0, B_1, \dots , and B_n , then H »

Clausal Form to Prolog

```
(student(A) ∨ ¬dorm_resident(A)) ∧  
(student(A) ∨ ¬takes(A,B) ∨ ¬class(B))
```

6. Use commutativity of \vee to move negated terms to the right of each clause.
7. Use $P_1 \vee \neg P_2 \equiv P_2 \Rightarrow P_1 \equiv P_1 \Leftarrow P_2$

```
(student(A) ← dorm_resident(A)) ∧  
(student(A) ← ¬(¬takes(A,B) ∨ ¬class(B)))
```

6. Move Horn clauses to Prolog:

```
student(A) :- dorm_resident(A).  
student(A) :- takes(A,B), class(B).
```

Skolemization

$\exists X [\text{takes}(X, \text{cs101}) \wedge \text{class_year}(X, 2)]$

introduce a Skolem constant to get rid of existential quantifier (\exists):

$\text{takes}(x, \text{cs101}) \wedge \text{class_year}(x, 2)$

$\forall X [\neg \text{dorm_resident}(X) \vee$
 $\quad \exists A [\text{campus_address_of}(X, A)]]$

introduce a Skolem function to get rid of existential quantifier (\exists):

$\forall X [\neg \text{dorm_resident}(X) \vee$
 $\quad \text{campus_address_of}(X, f(X))]$

Limitations

- If more than one non-negated (positive) term in a clause, then it cannot be moved to a Horn clause (which restricts clauses to only one head term).
- If zero non-negated (positive) terms, the same problem arises (Prolog's inability to prove logical negations).
- For example:
 - « every living thing is an animal or a plant »

$\text{animal}(X) \vee \text{plant}(X) \vee \neg \text{living}(X)$

$\text{animal}(X) \vee \text{plant}(X) \Leftarrow \text{living}(X)$

Exercises

72. What is the logical meaning of the second Skolemization example if we do not introduce a Skolem function?
73. Convert the following predicates into Conjunctive Normal Form, and if possible, into Horn clauses:
- a) $\forall C [\text{rainy}(C) \wedge \text{cold}(C) \Rightarrow \text{snowy}(C)]$
 - b) $\exists C [\neg \text{snowy}(C)]$
 - c) $\neg \exists C [\text{snowy}(C)]$
74. PLP Exercise 11.5 (pg 571).
75. PLP Exercise 11.6 (pg 571).