Functional Programming:
Lists, Pattern Matching, Recursive Programming
(CTM Sections 1.1-1.7, 3.2, 3.4.1-3.4.2, 4.7.2)

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Introduction to Oz

• An introduction to programming concepts
• Declarative variables
• Structured data (example: lists)
• Functions over lists
• Correctness and complexity
Variables

- Variables are short-cuts for values, they cannot be assigned more than once
  
  ```declare
  V = 9999*9999
  {Browse V*V}
  ```

- Variable identifiers: is what you type
- Store variable: is part of the memory system
- The `declare` statement creates a store variable and assigns its memory address to the identifier ‘V’ in the environment
Functions

• Compute the factorial function:
  \[ n! = 1 \times 2 \times \cdots \times (n - 1) \times n \]
  \[ 0! = 1 \]
  \[ n! = n \times (n - 1)! \text{ if } n > 0 \]

• Start with the mathematical definition

\[
\text{fun } \{\text{Fact } N\} \\
\text{declare} \\
\text{if } N==0 \text{ then 1 else } N*\{\text{Fact } N-1\} \text{ end} \\
\text{end}
\]

• Fact is declared in the environment

• Try large factorial \{Browse \{Fact 100\}\}
Factorial in Haskell

factorial :: Integer -> Integer
factorial 0 = 1
factorial n | n > 0 = n * factorial (n-1)
Composing functions

• Combinations of \( r \) items taken from \( n \).
• The number of subsets of size \( r \) taken from a set of size \( n \)

\[
\binom{n}{r} = \frac{n!}{r!(n-r)!}
\]

\[
\text{declare}
\]
\[
\text{fun \{Comb N R\}}
\]
\[
\{\text{Fact N}\} \div \{\text{Fact R}\} \ast \{\text{Fact N-R}\}
\]
\[
\text{end}
\]

• Example of functional abstraction
Structured data (lists)

- Calculate Pascal triangle
- Write a function that calculates the nth row as one structured value
- A list is a sequence of elements:
  \[
  \begin{bmatrix}
  1 & 4 & 6 & 4 & 1 \\
  \end{bmatrix}
  \]
- The empty list is written nil
- Lists are created by means of ”|” (cons)

```scheme
declare
H=1
T = [2 3 4 5]
{Browse H|T} % This will show [1 2 3 4 5]
```
• Taking lists apart (selecting components)
• A cons has two components: a head, and a tail

\[
\text{declare } L = [5 \ 6 \ 7 \ 8]
\]

L.1 gives 5
L.2 give [6 7 8]
Pattern matching

• Another way to take a list apart is by use of pattern matching with a case instruction

```
case L of H|T then {Browse H} {Browse T}
    else {Browse ‘empty list’ } 
end
```
Functions over lists

- Compute the function \{\text{Pascal N}\}
- Takes an integer \(N\), and returns the \(N\)th row of a Pascal triangle as a list
  1. For row 1, the result is [1]
  2. For row \(N\), shift to left row \(N-1\) and shift to the right row \(N-1\)
  3. Align and add the shifted rows element-wise to get row \(N\)

\[
\begin{array}{cccccc}
1 & & & & & \\
& 1 & 2 & 1 & & \\
& & 1 & 3 & 3 & 1 \\
& (0) & 1 & 3 & 3 & 1 & (0) \\
\end{array}
\]

Shift right [0 1 3 3 1]

Shift left [1 3 3 1 0]
Functions over lists (2)

```plaintext
declare
fun {Pascal N}
  if N==1 then [1]
  else
    {AddList
      {ShiftLeft {Pascal N-1}}
      {ShiftRight {Pascal N-1}}}
  end
end
```
Functions over lists (3)

fun \{\text{ShiftLeft} L\} \\
\begin{align*}
\text{case } L \text{ of } & H|T \text{ then} \\
& H|\{\text{ShiftLeft} T\} \\
\text{else } & [0] \\
\text{end} \\
\text{end}
\end{align*}

fun \{\text{ShiftRight} L\} \ 0|L \text{ end}

fun \{\text{AddList} L1 L2\} \\
\begin{align*}
\text{case } L1 \text{ of } & H1|T1 \text{ then} \\
& \text{case } L2 \text{ of } H2|T2 \text{ then} \\
& \quad H1+H2|\{\text{AddList} T1 T2\} \\
& \text{end} \\
\text{else } & \text{nil} \text{ end} \\
\text{end}
\end{align*}
Top-down program development

- Understand how to solve the problem by hand
- Try to solve the task by decomposing it to simpler tasks
- Devise the main function (main task) in terms of suitable auxiliary functions (subtasks) that simplify the solution (ShiftLeft, ShiftRight and AddList)
- Complete the solution by writing the auxiliary functions
- Test your program bottom-up: auxiliary functions first.
Is your program correct?

• “A program is correct when it does what we would like it to do”

• In general we need to reason about the program:
  • **Semantics for the language**: a precise model of the operations of the programming language
  • **Program specification**: a definition of the output in terms of the input (usually a mathematical function or relation)
  • Use mathematical techniques to reason about the program, using programming language semantics
Mathematical induction

• Select one or more inputs to the function
• Show the program is correct for the simple cases (base cases)
• Show that if the program is correct for a given case, it is then correct for the next case.
• For natural numbers, the base case is either 0 or 1, and for any number n the next case is n+1
• For lists, the base case is nil, or a list with one or a few elements, and for any list T the next case is H|T
Correctness of factorial

fun \{\text{Fact } N}\}
   if \ N==0 \ then \ 1 \ else \ N*\{\text{Fact } N-1\} \ end
end

1 \times 2 \times \cdots \times (n-1) \times n \quad \text{Fact}(n-1)

• Base Case \(N=0\): \{\text{Fact } 0\} \text{ returns } 1
• Inductive Case \(N>0\): \{\text{Fact } N\} \text{ returns } N*\{\text{Fact } N-1\} \text{ assume } \{\text{Fact } N-1\} \text{ is correct, from the spec we see that } \{\text{Fact } N\} \text{ is } N*\{\text{Fact } N-1\}
Complexity

• Pascal runs very slow, try {Pascal 24}
• {Pascal 20} calls: {Pascal 19} twice, {Pascal 18} four times, {Pascal 17} eight times, ..., {Pascal 1} $2^{19}$ times
• Execution time of a program up to a constant factor is called the program’s time complexity.
• Time complexity of {Pascal N} is proportional to $2^N$ (exponential)
• Programs with exponential time complexity are impractical
Faster Pascal

- Introduce a local variable L
- Compute \{FastPascal N-1\} only once
- Try with 30 rows.
- FastPascal is called N times, each time a list on the average of size N/2 is processed
- The time complexity is proportional to \(N^2\) (polynomial)
- Low order polynomial programs are practical.

```pascal
fun {FastPascal N}
  if N==1 then [1]
  else
    local L in
    L={FastPascal N-1}
    {AddList {ShiftLeft L} {ShiftRight L}}
  end
end
end
```
Iterative computation

- An iterative computation is one whose execution stack is bounded by a constant, independent of the length of the computation.
- Iterative computation starts with an initial state $S_0$, and transforms the state in a number of steps until a final state $S_{final}$ is reached:

$$S_0 \rightarrow S_1 \rightarrow \ldots \rightarrow S_{final}$$
The general scheme

fun \{Iterate \( S_i \)}
    if \{IsDone \( S_i \)} then \( S_i \)
    else \( S_{i+1} \) in
        \( S_{i+1} = \{Transform \ S_i \} \)
        \{Iterate \( S_{i+1} \} \)
    end
end

• \textit{IsDone} and \textit{Transform} are problem dependent
The computation model

- STACK : [ R={Iterate $S_0$} ]

- STACK : [ $S_1 = \{\text{Transform } S_0\}$,  
  R={Iterate $S_1$} ]

- STACK : [ R={Iterate $S_i$} ]

- STACK : [ $S_{i+1} = \{\text{Transform } S_i\}$,  
  R={Iterate $S_{i+1}$} ]

- STACK : [ R={Iterate $S_{i+1}$} ]
Newton’s method for the square root of a positive real number

- Given a real number $x$, start with a guess $g$, and improve this guess iteratively until it is accurate enough.

- The improved guess $g'$ is the average of $g$ and $x/g$:

  $$g' = (g + x/g) / 2$$

  $$\varepsilon = g - \sqrt{x}$$

  $$\varepsilon' = g' - \sqrt{x}$$

  For $g'$ to be a better guess than $g$: $\varepsilon' < \varepsilon$

  $$\varepsilon' = g' - \sqrt{x} = (g + x/g) / 2 - \sqrt{x} = \varepsilon^2 / 2g$$

  i.e. $\varepsilon^2 / 2g < \varepsilon$, $\varepsilon / 2g < 1$

  i.e. $\varepsilon < 2g$, $g - \sqrt{x} < 2g$, $0 < g + \sqrt{x}$
Newton’s method for the square root of a positive real number

- Given a real number $x$, start with a guess $g$, and improve this guess iteratively until it is accurate enough.
- The improved guess $g'$ is the average of $g$ and $x/g$.
- Accurate enough is defined as:

\[ \frac{|x - g^2|}{x} < 0.00001 \]
fun {SqrtIter Guess X}
    if {GoodEnough Guess X} then Guess
    else
        Guess1 = {Improve Guess X} in
        {SqrtIter Guess1 X}
    end
end

• Compare to the general scheme:
  – The state is the pair Guess and X
  – *IsDone* is implemented by the procedure *GoodEnough*
  – *Transform* is implemented by the procedure *Improve*
The program version 1

fun {Sqrt X}
   Guess = 1.0
in {SqrtIter Guess X}
end

fun {SqrtIter Guess X}
   if {GoodEnough Guess X} then
      Guess
   else
      {SqrtIter {Improve Guess X} X}
   end
end

fun {Improve Guess X}
   (Guess + X/Guess)/2.0
end

fun {GoodEnough Guess X}
   {Abs X - Guess*Guess}/X < 0.00001
end
Using local procedures

- The main procedure Sqrt uses the helper procedures SqrtIter, GoodEnough, Improve, and Abs
- SqrtIter is only needed inside Sqrt
- GoodEnough and Improve are only needed inside SqrtIter
- Abs (absolute value) is a general utility
- The general idea is that helper procedures should not be visible globally, but only locally
local
fun {SqrtIter Guess X}
  if {GoodEnough Guess X} then Guess
  else {SqrtIter {Improve Guess X} X} end
end
fun {Improve Guess X}
  (Guess + X/Guess)/2.0
end
fun {GoodEnough Guess X}
  {Abs X - Guess*Guess}/X < 0.000001
end
in
fun {Sqrt X}
  Guess = 1.0
  in {SqrtIter Guess X} end
end
Sqrt version 3

- Define GoodEnough and Improve inside SqrtIter

```plaintext
local
fun {SqrtIter Guess X}
  fun {Improve}
    (Guess + X/Guess)/2.0
  end
  fun {GoodEnough}
    {Abs X - Guess*Guess}/X < 0.000001
  end
  in
    if {GoodEnough} then Guess
    else {SqrtIter {Improve} X} end
  end
in fun {Sqrt X}
  Guess = 1.0 in
  {SqrtIter Guess X}
end
end
```
Sqrt version 3

- Define GoodEnough and Improve inside SqrtIter

local
  fun {SqrtIter Guess X}
    fun {Improve}
      (Guess + X/Guess)/2.0
    end
    fun {GoodEnough}
      {Abs X - Guess*Guess}/X < 0.000001
    end
  in
    if {GoodEnough} then Guess
    else {SqrtIter {Improve} X} end
  end
in fun {Sqrt X}
  Guess = 1.0 in
  {SqrtIter Guess X}
end end

The program has a single drawback: on each iteration two procedure values are created, one for Improve and one for GoodEnough
fun {Sqrt X}
  fun {Improve Guess}
    (Guess + X/Guess)/2.0
  end
  fun {GoodEnough Guess}
    {Abs X - Guess*Guess}/X < 0.000001
  end
  fun {SqrtIter Guess}
    if {GoodEnough Guess} then Guess
    else {SqrtIter {Improve Guess} } end
  end
  Guess = 1.0
in {SqrtIter Guess}
end
From a general scheme
to a control abstraction (1)

fun \{\text{Iterate } S_i\} 
  if \{\text{IsDone } S_i\} \text{ then } S_i 
  else \text{ if } S_{i+1} \text{ in } 
          S_{i+1} = \{\text{Transform } S_i\} 
          \{\text{Iterate } S_{i+1}\} 
  end 
end 
end 

- \text{IsDone and Transform are problem dependent}
From a general scheme to a control abstraction (2)

fun \{Iterate \text{S} \ IsDone \text{Transform}\}
  if \{IsDone \text{S}\} then \text{S}
  else \text{S1} in
    \text{S1} = \{\text{Transform} \text{S}\}
    \{\text{Iterate \text{S1} IsDone Transform}\}
  end
end
end

fun \{Iterate \text{S}_i\}
  if \{IsDone \text{S}_i\} then \text{S}_i
  else \text{S}_{i+1} in
    \text{S}_{i+1} = \{\text{Transform} \text{S}_i\}
    \{\text{Iterate \text{S}_{i+1}}\}
  end
end
Sqrt using the Iterate abstraction

fun {Sqrt X}
  fun {Improve Guess}
    (Guess + X/Guess)/2.0
  end
  fun {GoodEnough Guess}
    {Abs X - Guess*Guess}/X < 0.000001
  end
  Guess = 1.0
in
  {Iterate Guess GoodEnough Improve}
end
Sqrt using the control abstraction

fun {Sqrt X}
  {Iterate
    1.0
    fun {$ G} {Abs X - G*G}/X < 0.000001 end
    fun {$ G} (G + X/G)/2.0 end
  }
end

Iterate could become a linguistic abstraction
Sqrt using Iterate in Haskell

\[ \text{iterate}' \ s \ \text{isDone} \ \text{transform} = \]
\[ \quad \text{if isDone} \ s \ \text{then} \ s \]
\[ \quad \text{else let} \ s_1 = \text{transform} \ s \ \text{in} \]
\[ \quad \quad \text{iterate}' \ s_1 \ \text{isDone} \ \text{transform} \]

\[ \text{sqrt}' \ x = \text{iterate}' \ 1.0 \ \text{goodEnough} \ \text{improve} \]
\[ \quad \text{where} \ \text{goodEnough} = \lambda g \rightarrow (\text{abs} \ (x - g\times g))/x < 0.00001 \]
\[ \quad \quad \text{improve} = \lambda g \rightarrow (g + x/g)/2.0 \]
Sqrt in Haskell

sqrt x = head (dropWhile (not . goodEnough) sqrtGuesses)

where

  goodEnough guess = (abs (x – guess*guess))/x < 0.00001

  improve guess = (guess + x/guess)/2.0

sqrtGuesses = 1:(map improve sqrtGuesses)

This sqrt example uses infinite lists enabled by lazy evaluation, and the map control abstraction.
12. Prove the correctness of AddList and ShiftLeft.
13. Prove that the alternative version of Pascal triangle (not using ShiftLeft) is correct. Make AddList and OpList commutative.
14. Modify the Pascal function to use local functions for AddList, ShiftLeft, ShiftRight. Think about the abstraction and efficiency tradeoffs.
15. CTM Exercise 3.10.2 (page 230)
16. CTM Exercise 3.10.3 (page 230)
17. Develop a control abstraction for iterating over a list of elements.