Approximation Algorithms for the Firefighter Problem

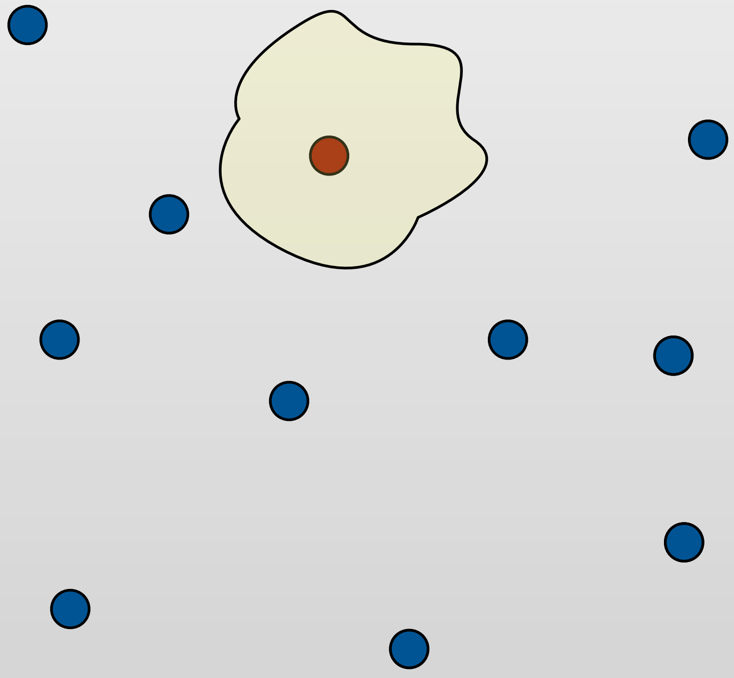
Cuts over time and Submodularity

Elliot Anshelevich, Ameya Hate
Rensselaer Polytechnic Institute (RPI)

Deeparnab Chakrabarty, Chaitanya Swamy
University of Waterloo

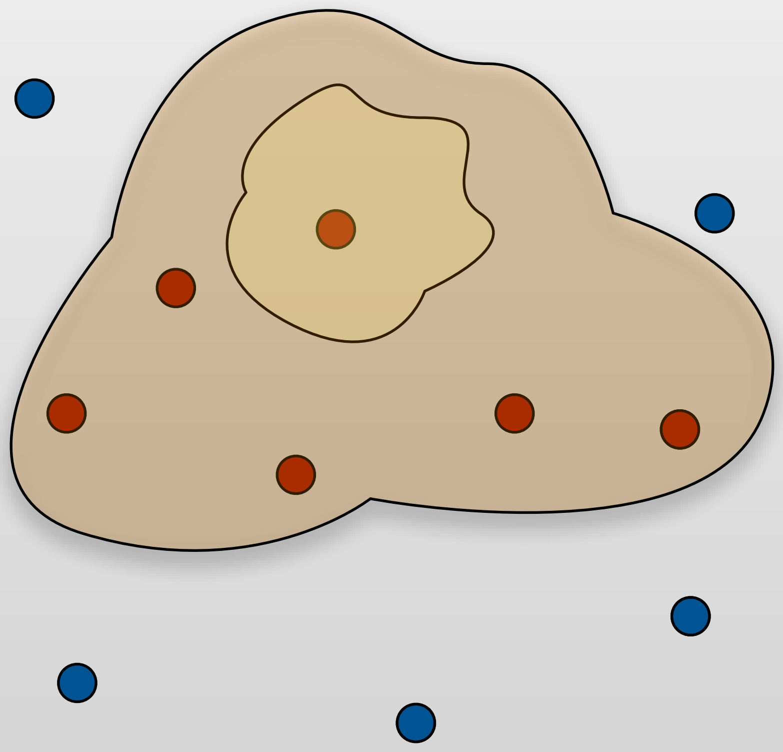
DIFFUSIVE NETWORK PROCESS

- Diffusive processes such as ideas or disease spreading through population
- Social networks modeled using graph theory
- Graph theory concepts can be used to study the spread of these entities
- Useful in a variety of areas ranging from epidemic spread to viral marketing



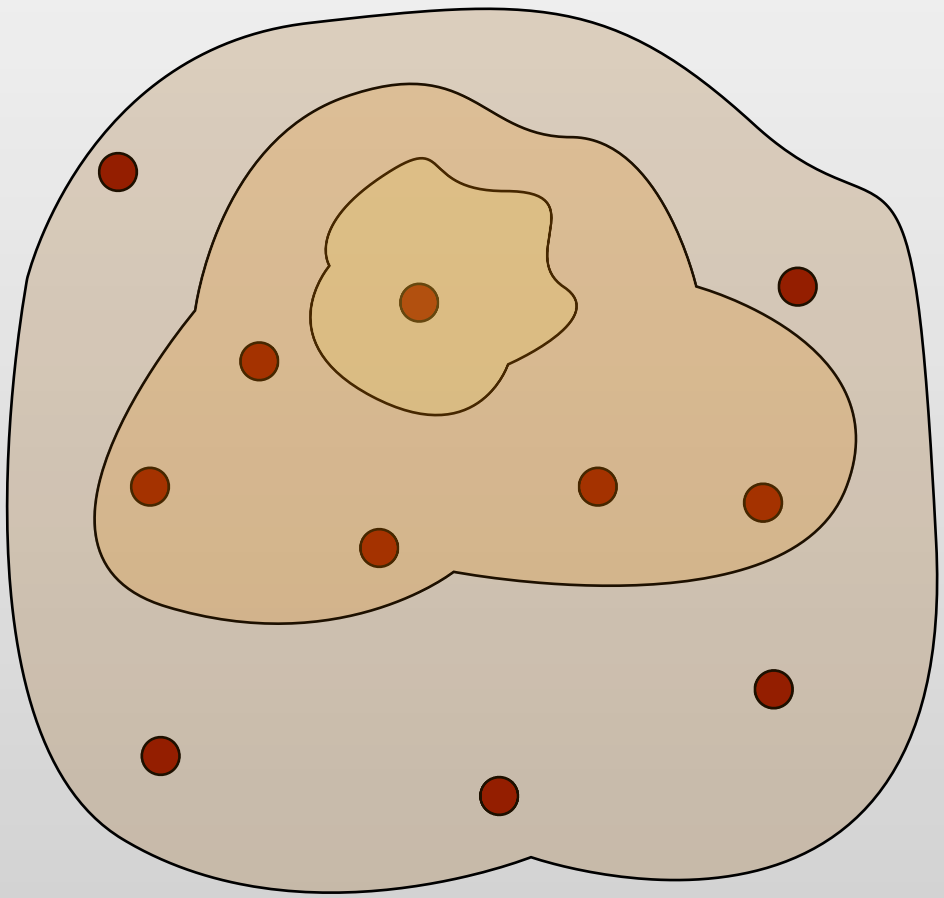
DIFFUSIVE NETWORK PROCESS

- Diffusive processes such as ideas or disease spreading through population
- Social networks modeled using graph theory
- Graph theory concepts can be used to study the spread of these entities
- Useful in a variety of areas ranging from epidemic spread to viral marketing



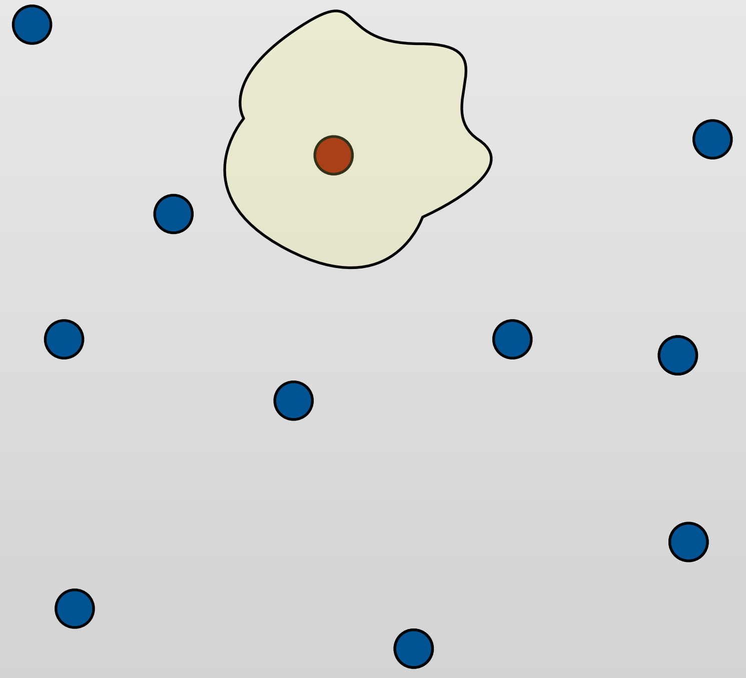
DIFFUSIVE NETWORK PROCESS

- Diffusive processes such as ideas or disease spreading through population
- Social networks modeled using graph theory
- Graph theory concepts can be used to study the spread of these entities
- Useful in a variety of areas ranging from epidemic spread to viral marketing



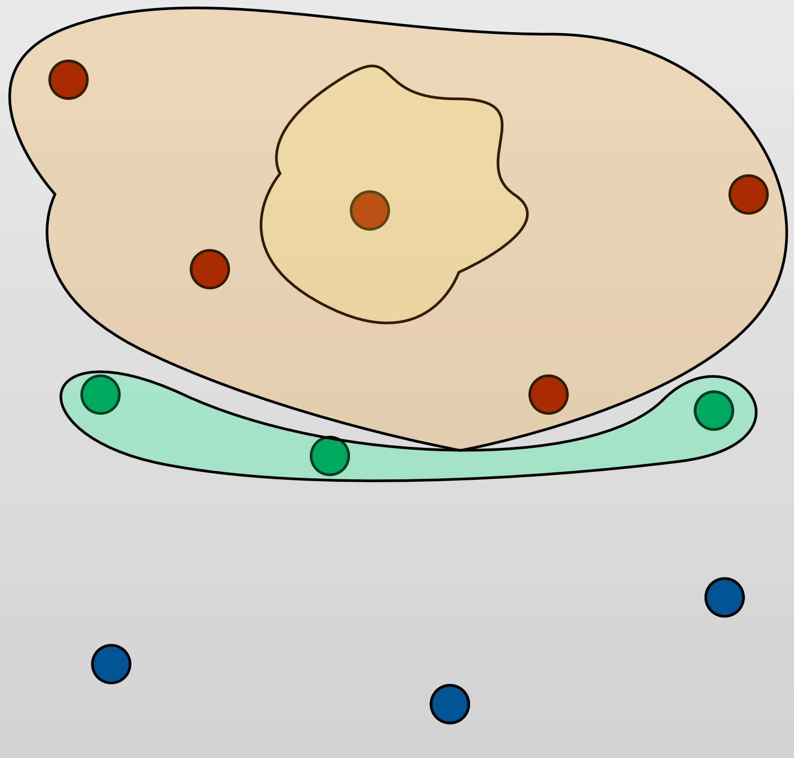
CONTAINING DIFFUSION

- This paper deals with inhibiting spread of infection through targeted vaccination
- We give worst case guarantees over all possible networks: Graph is part of input
- Our goal is to vaccinate parts of graphs once the epidemic has begun



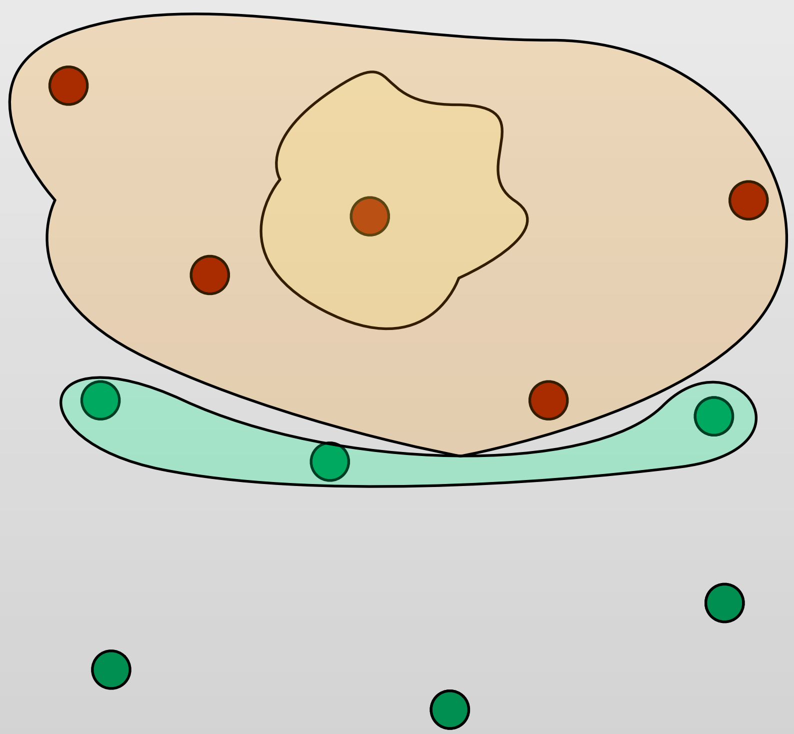
CONTAINING DIFFUSION

- This paper deals with inhibiting spread of infection through targeted vaccination
- We give worst case guarantees over all possible networks: Graph is part of input
- Our goal is to vaccinate parts of graphs once the epidemic has begun

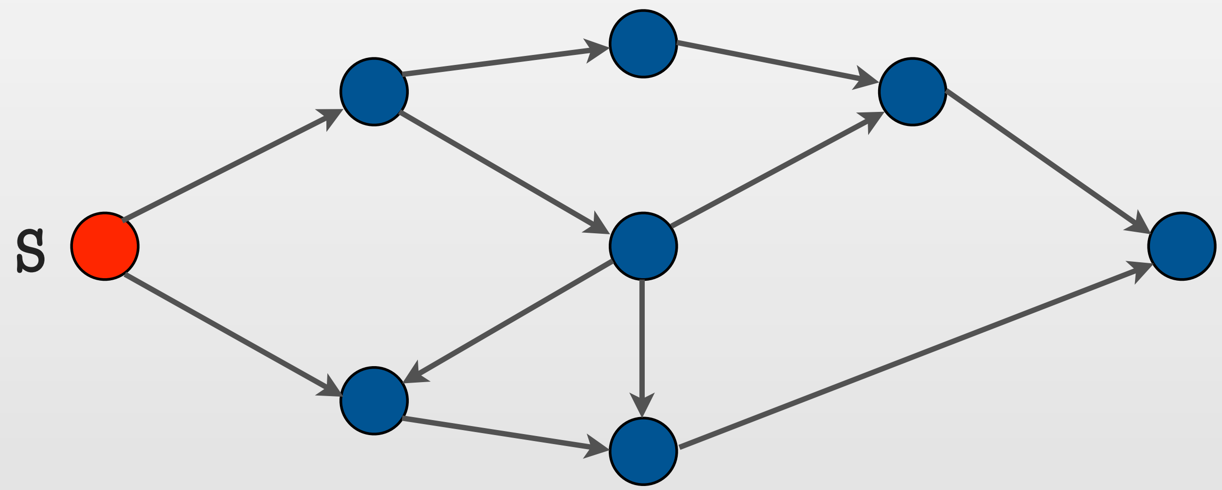


CONTAINING DIFFUSION

- This paper deals with inhibiting spread of infection through targeted vaccination
- We give worst case guarantees over all possible networks: Graph is part of input
- Our goal is to vaccinate parts of graphs once the epidemic has begun

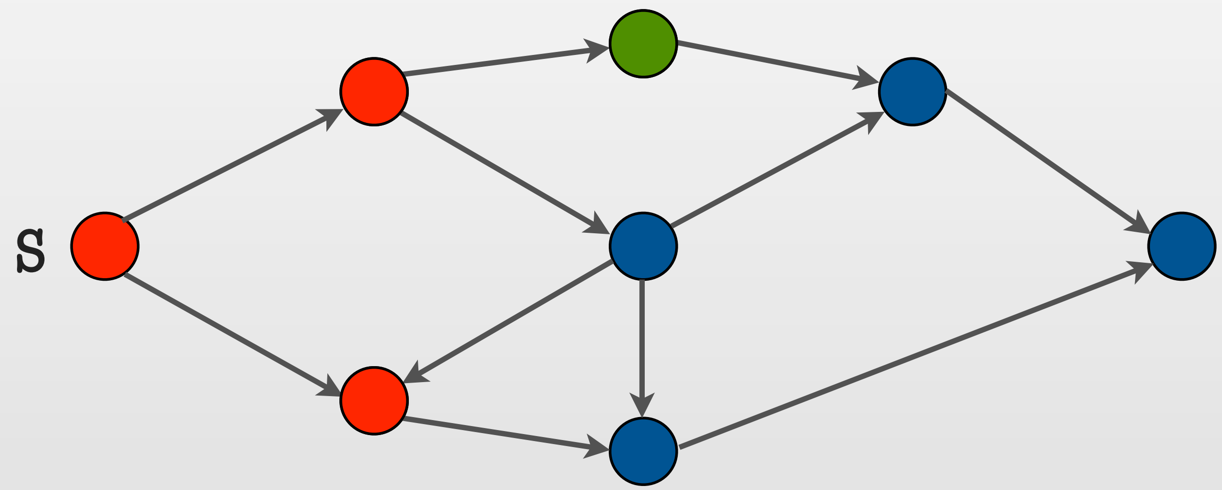


THE MODEL



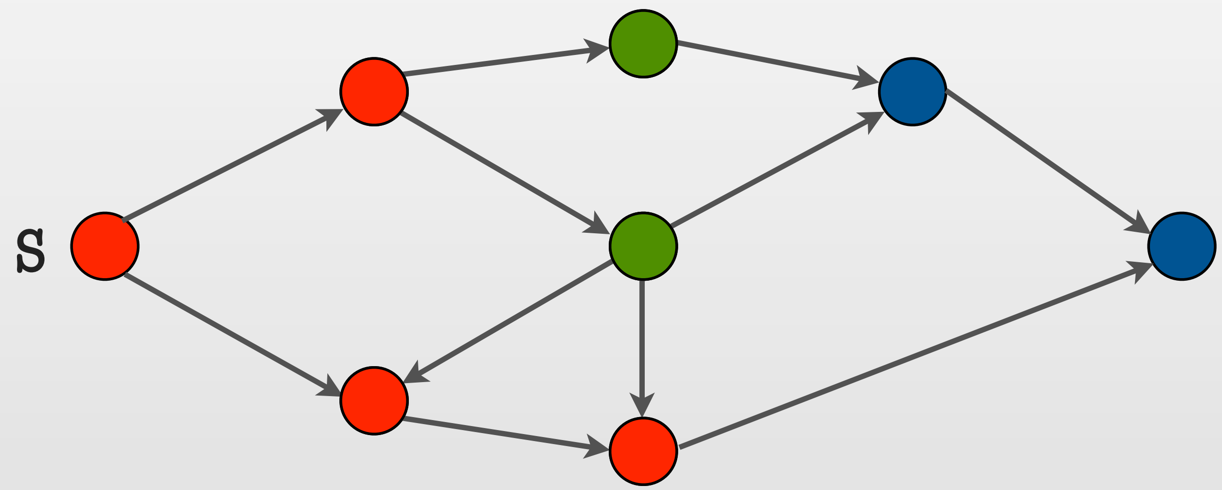
- Graph `G(V,E)` and vertex `s` where infection begins
- Vertex states: vulnerable, infected, vaccinated
- Budget `B` for vaccination per step
- A vaccination strategy w.r.t. budget `B`

THE MODEL



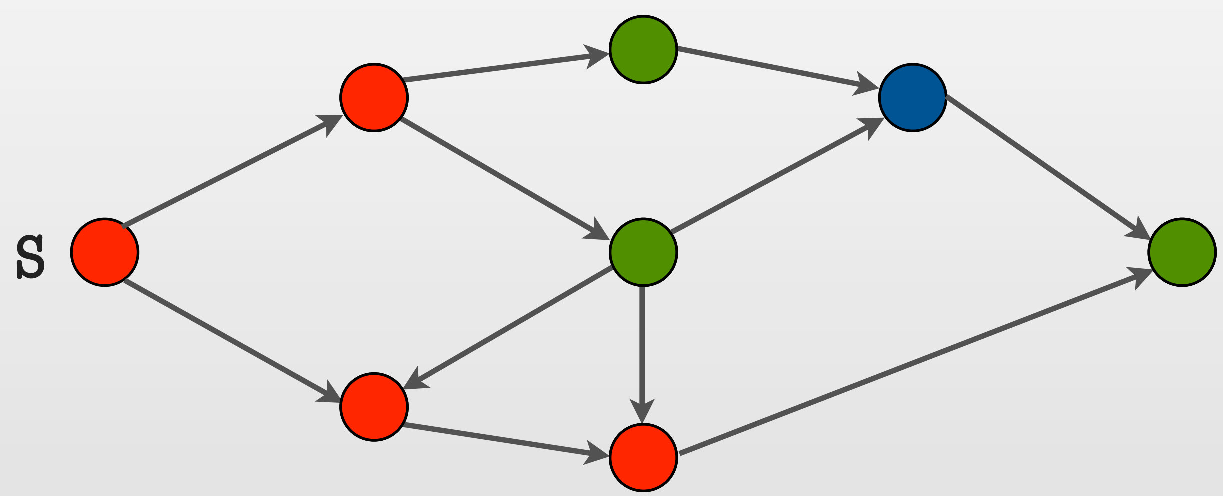
- Graph `G(V,E)` and vertex `s` where infection begins
- Vertex states: vulnerable, infected, vaccinated
- Budget `B` for vaccination per step
- A vaccination strategy w.r.t. budget `B`

THE MODEL



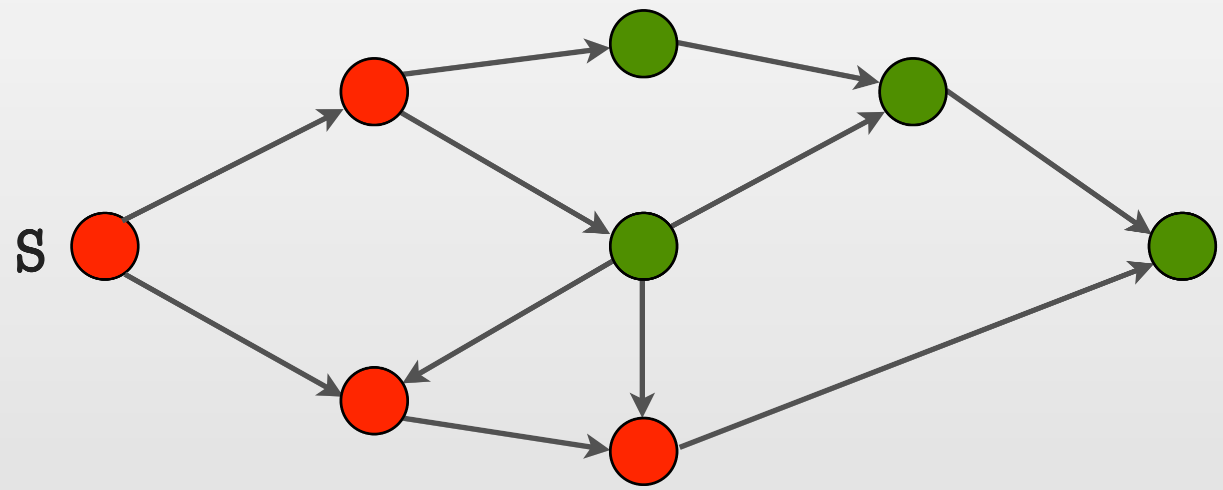
- Graph G(V,E) and vertex s where infection begins
- Vertex states: vulnerable, infected, vaccinated
- Budget B for vaccination per step
- A vaccination strategy w.r.t. budget B

THE MODEL



- Graph G(V,E) and vertex s where infection begins
- Vertex states: vulnerable, infected, vaccinated
- Budget B for vaccination per step
- A vaccination strategy w.r.t. budget B

THE MODEL



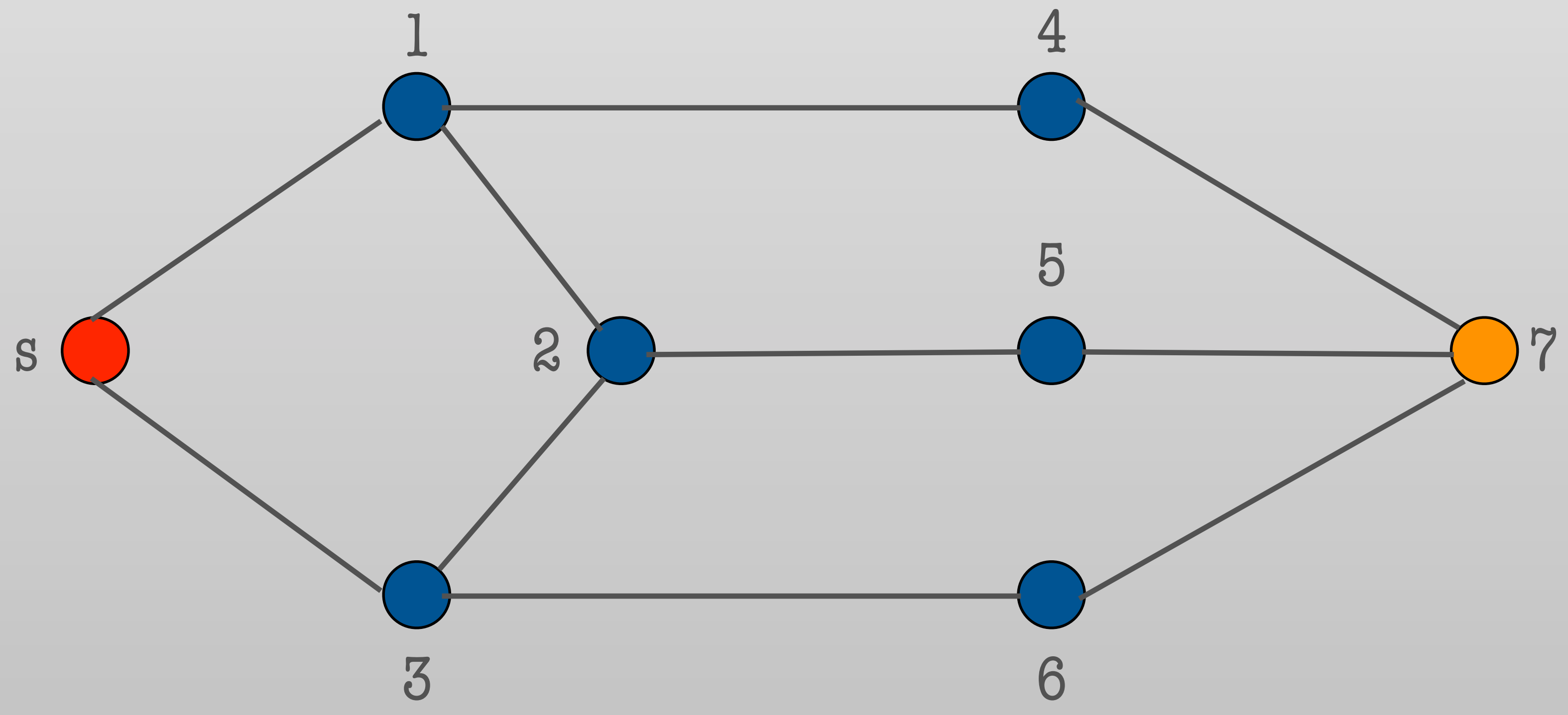
- Graph G(V,E) and vertex s where infection begins
- Vertex states: vulnerable, infected, vaccinated
- Budget B for vaccination per step
- A vaccination strategy w.r.t. budget B

STATIC VS DYNAMIC CUTS

- Notion of 'cut over time' as opposed to s-t cut
- Difficulty arises due to temporal component of the problem
  - Cut near or cut far?

STATIC VS DYNAMIC CUTS

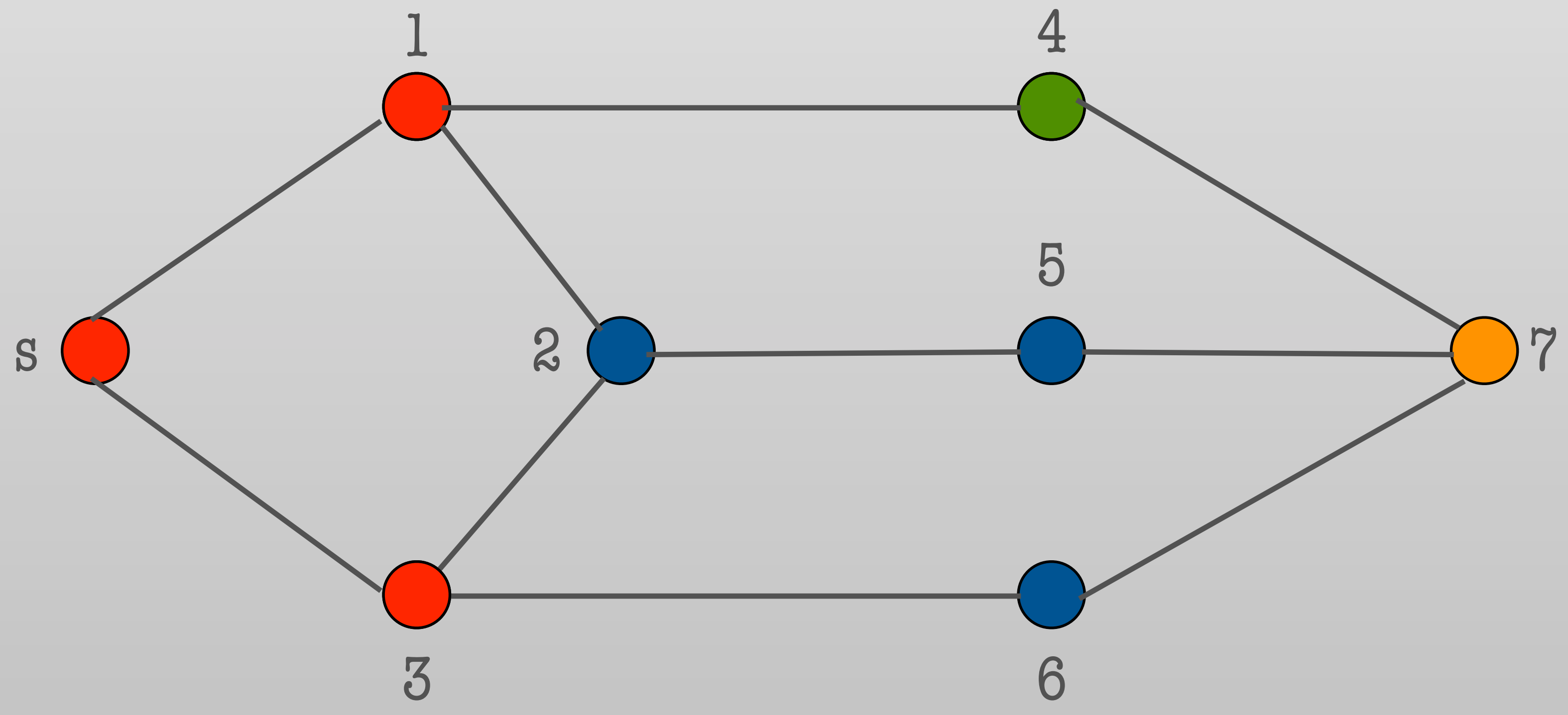
- Notion of 'cut over time' as opposed to s-t cut
- Difficulty arises due to temporal component of the problem
  - Cut near or cut far?



Example with
B = 1

STATIC VS DYNAMIC CUTS

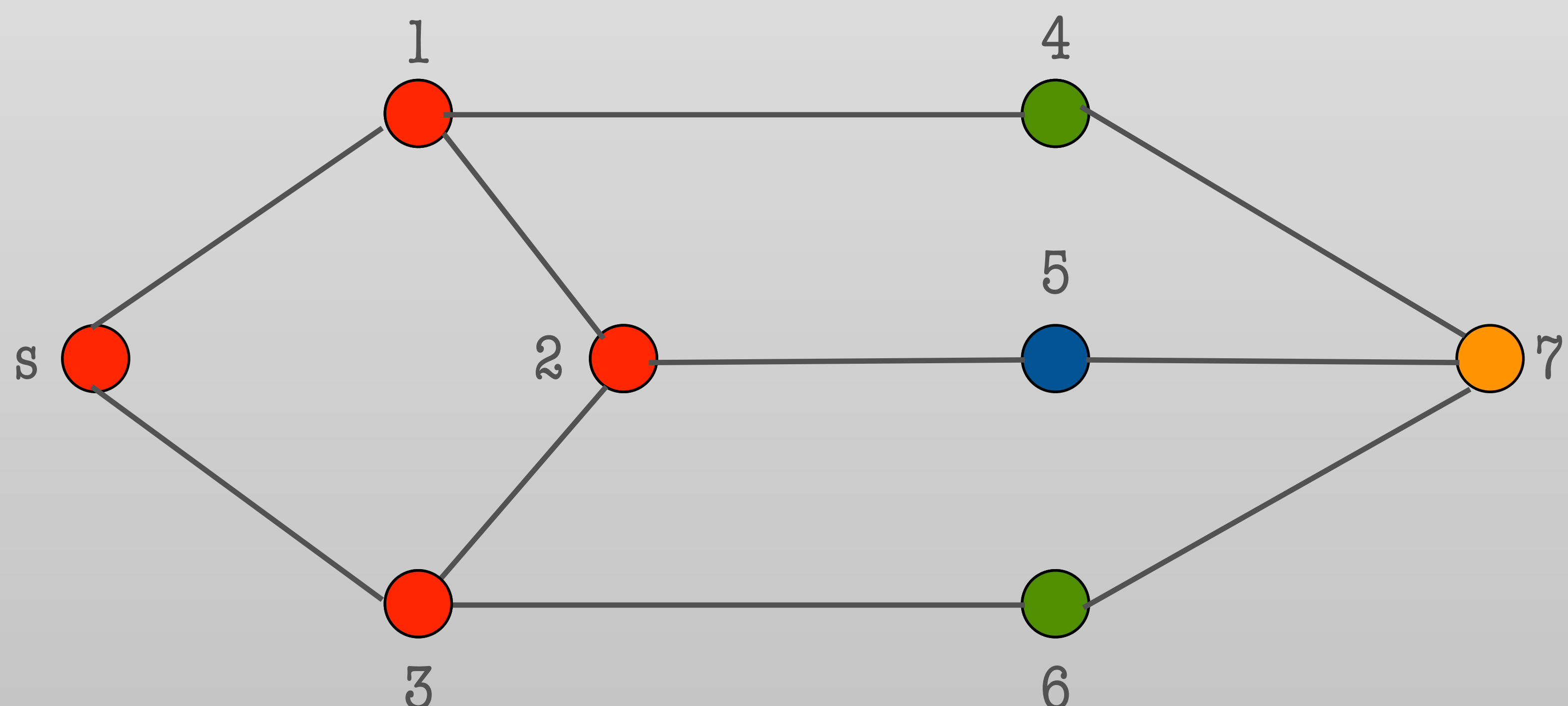
- Notion of 'cut over time' as opposed to s-t cut
- Difficulty arises due to temporal component of the problem
  - Cut near or cut far?



Example with B = 1

STATIC VS DYNAMIC CUTS

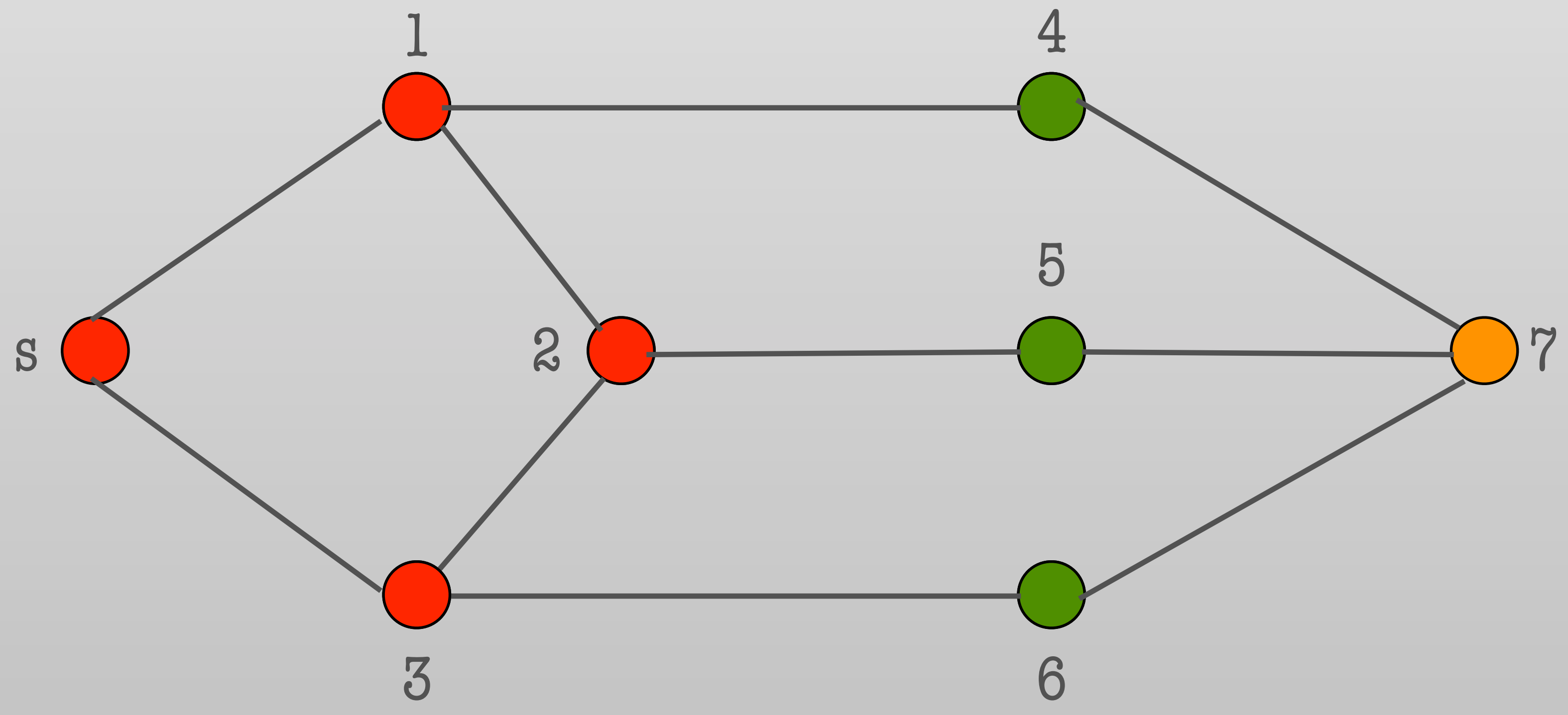
- Notion of 'cut over time' as opposed to s-t cut
- Difficulty arises due to temporal component of the problem
  - Cut near or cut far?



Example with
B = 1

STATIC VS DYNAMIC CUTS

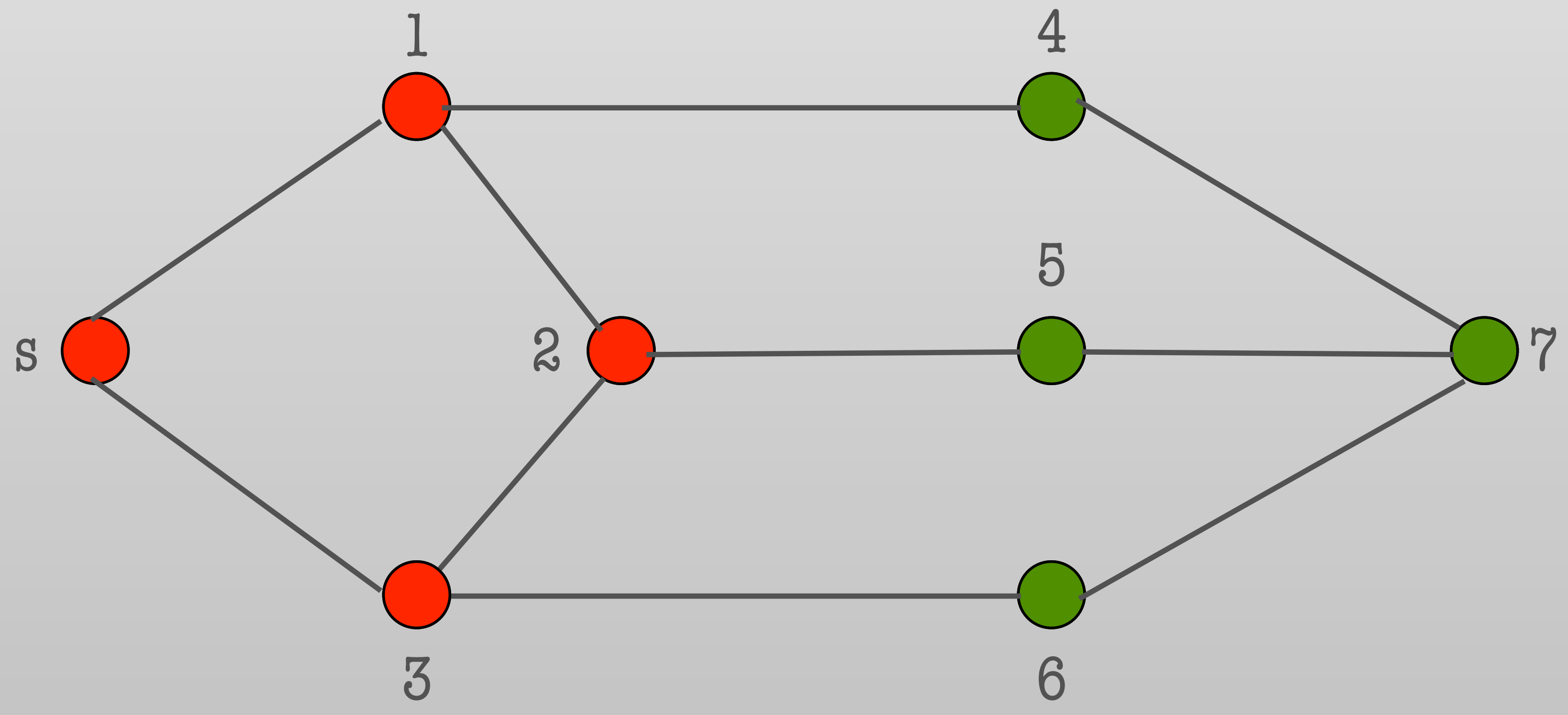
- Notion of 'cut over time' as opposed to s-t cut
- Difficulty arises due to temporal component of the problem
  - Cut near or cut far?



Example with
B = 1

STATIC VS DYNAMIC CUTS

- Notion of 'cut over time' as opposed to s-t cut
- Difficulty arises due to temporal component of the problem
    - Cut near or cut far?



Example with
B = 1

ANOTHER MODEL

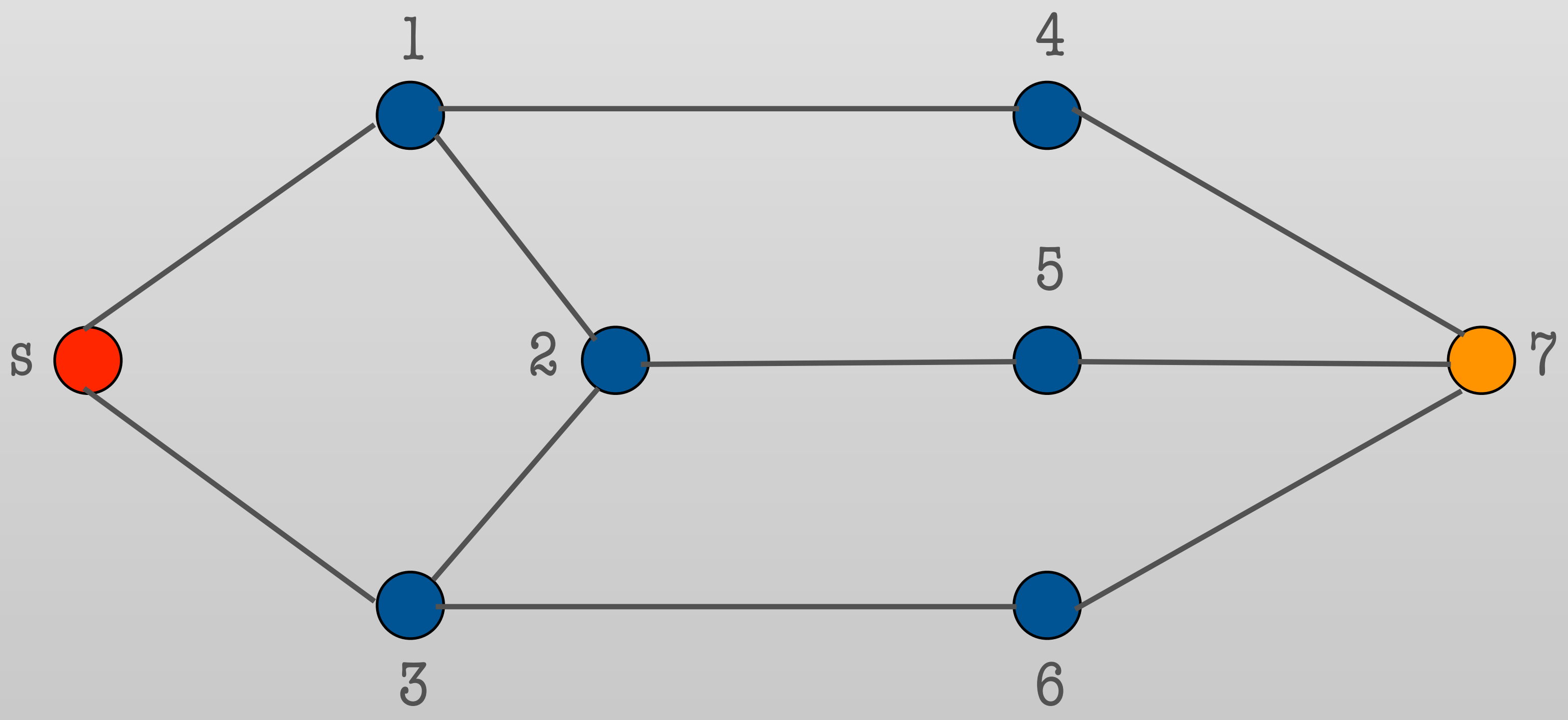
- A variant: Spreading Model
- Vaccines can be infectious processes as well: competing ideas in a social network
- At every step vaccine spreads to vulnerable neighbouring vertices

ANOTHER MODEL

- A variant: Spreading Model

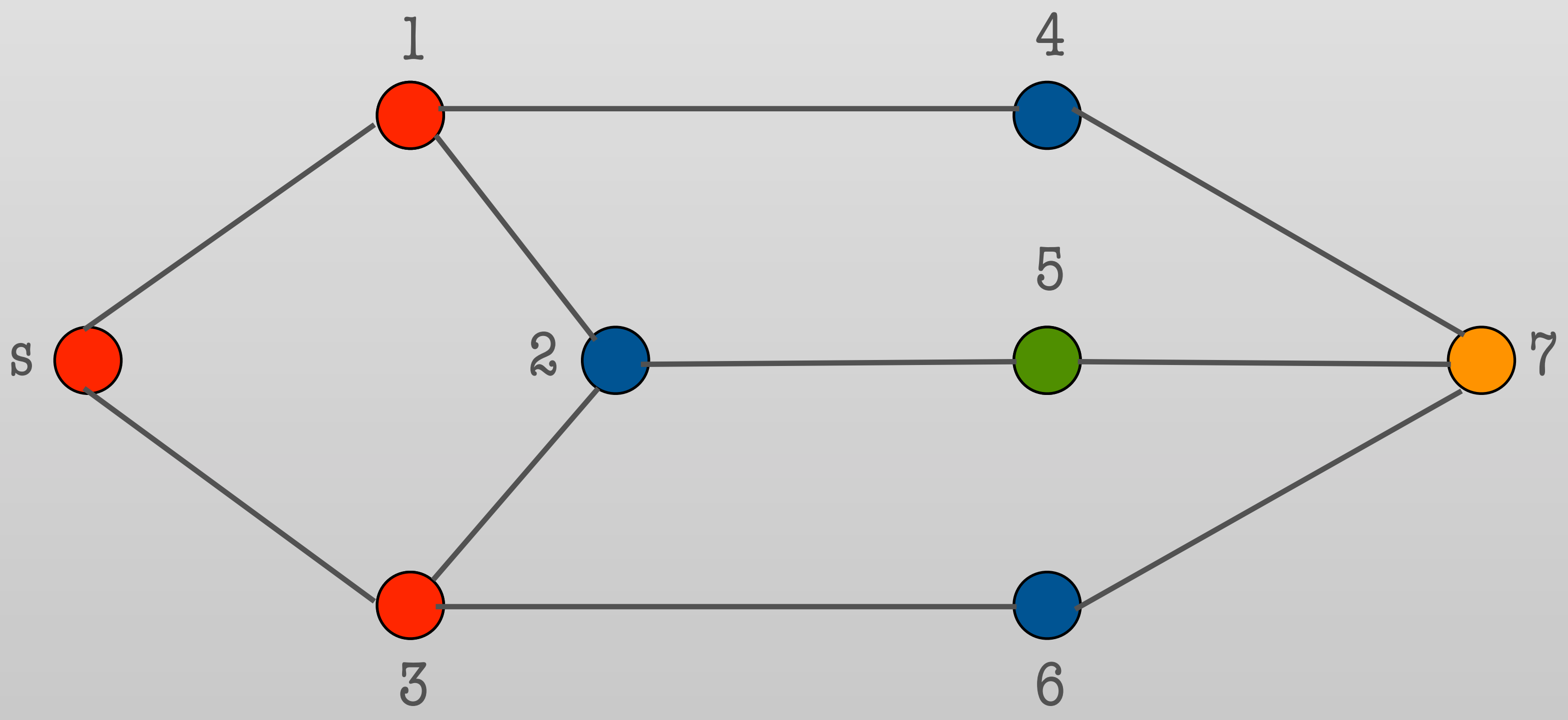
- Vaccines can be infectious processes as well: competing ideas in a social network

- At every step vaccine spreads to vulnerable neighbouring vertices



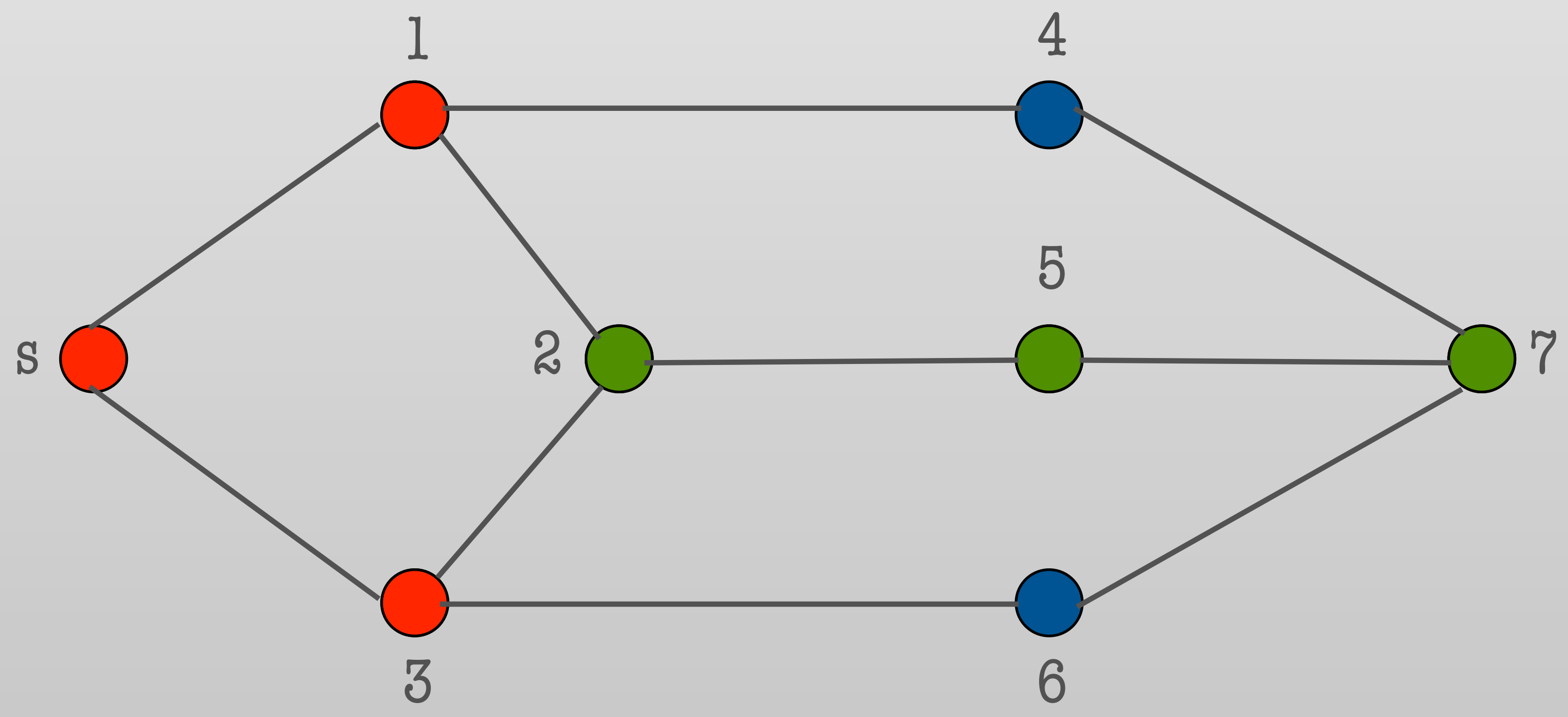
ANOTHER MODEL

- A variant: Spreading Model
- Vaccines can be infectious processes as well: competing ideas in a social network
- At every step vaccine spreads to vulnerable neighbouring vertices



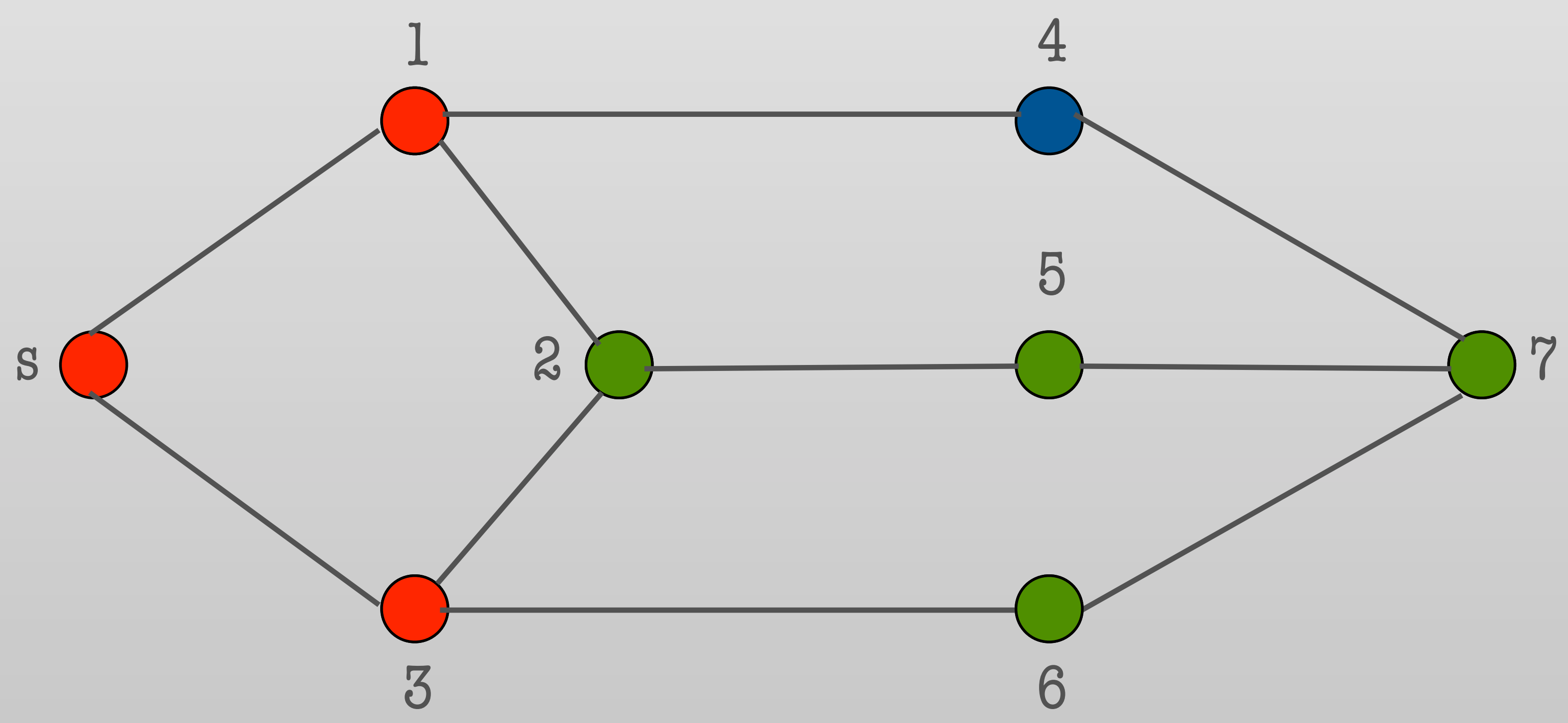
ANOTHER MODEL

- A variant: Spreading Model
- Vaccines can be infectious processes as well: competing ideas in a social network
- At every step vaccine spreads to vulnerable neighbouring vertices



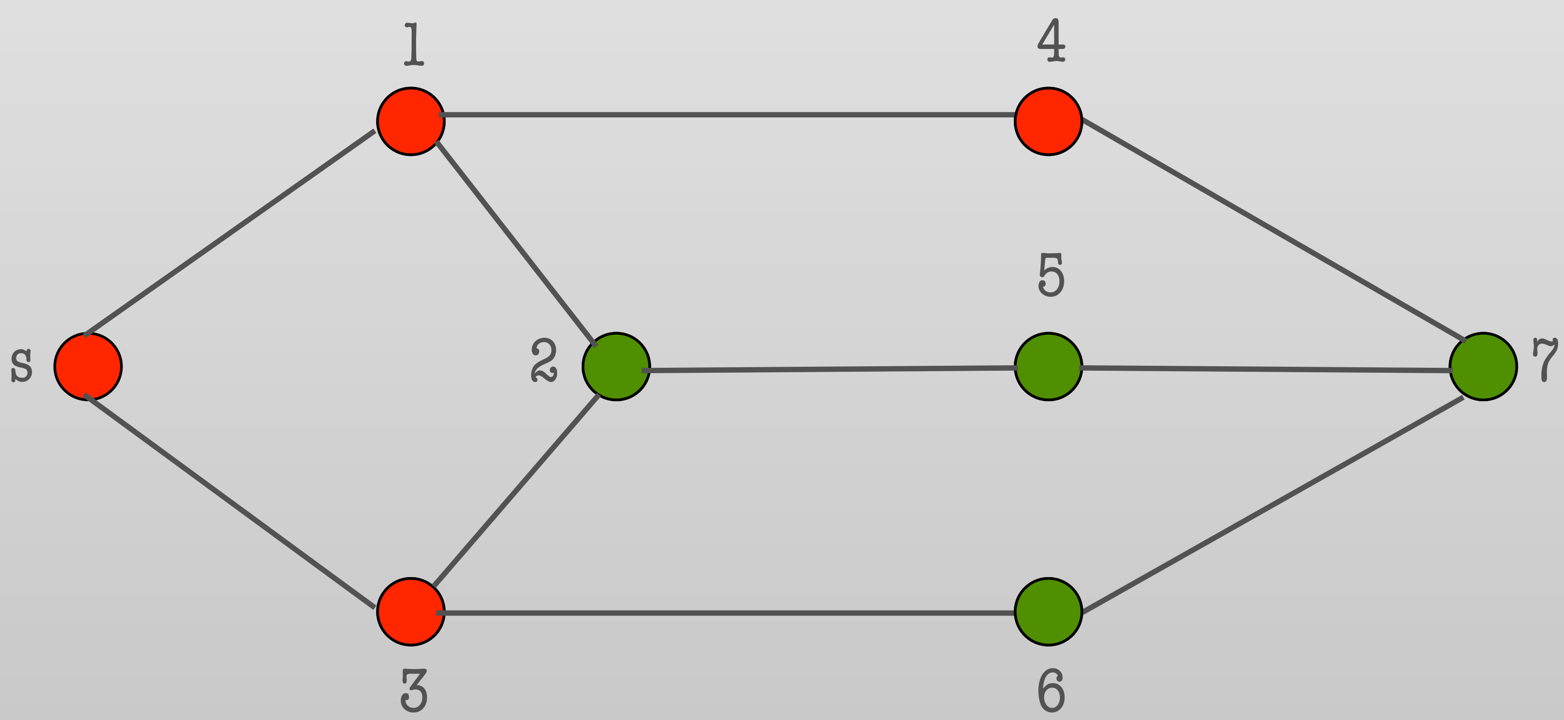
ANOTHER MODEL

- A variant: Spreading Model
- Vaccines can be infectious processes as well: competing ideas in a social network
- At every step vaccine spreads to vulnerable neighbouring vertices



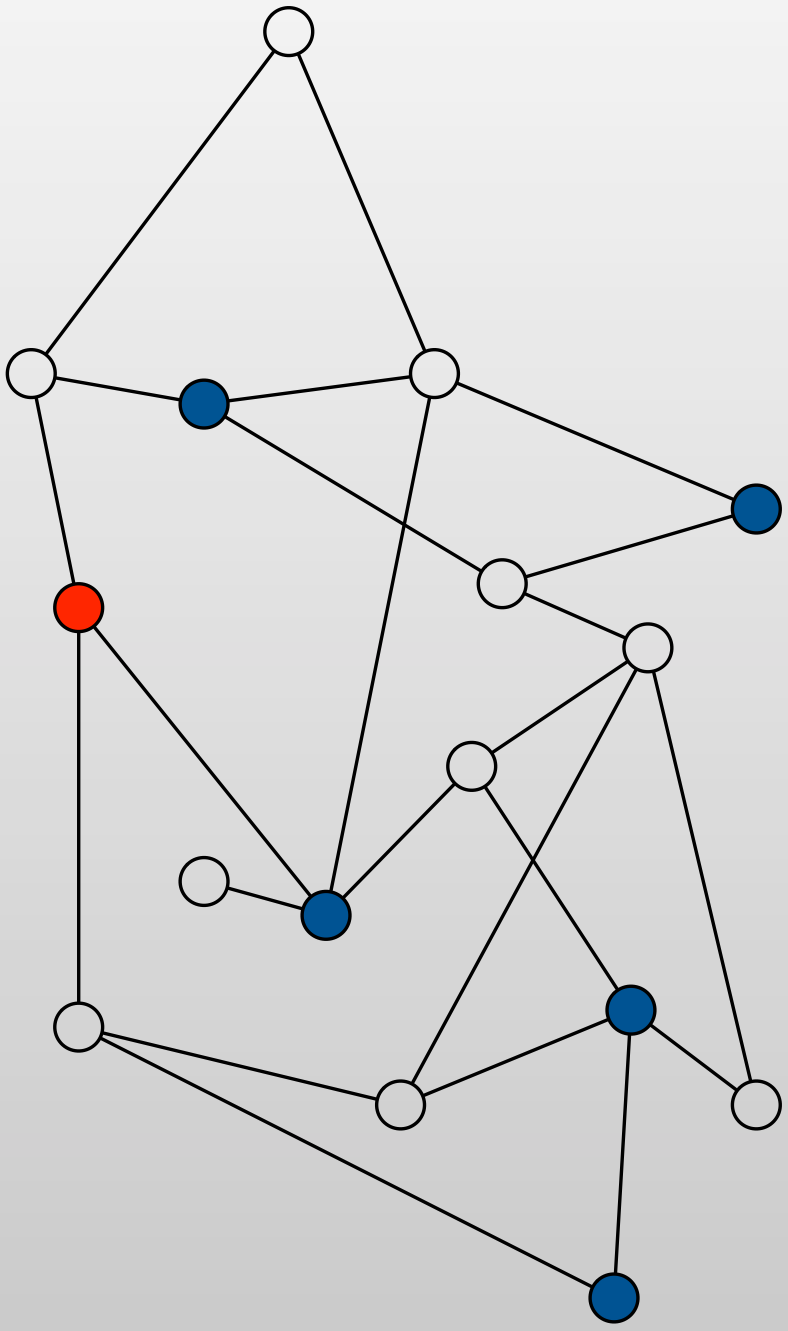
ANOTHER MODEL

- A variant: Spreading Model
- Vaccines can be infectious processes as well: competing ideas in a social network
- At every step vaccine spreads to vulnerable neighbouring vertices



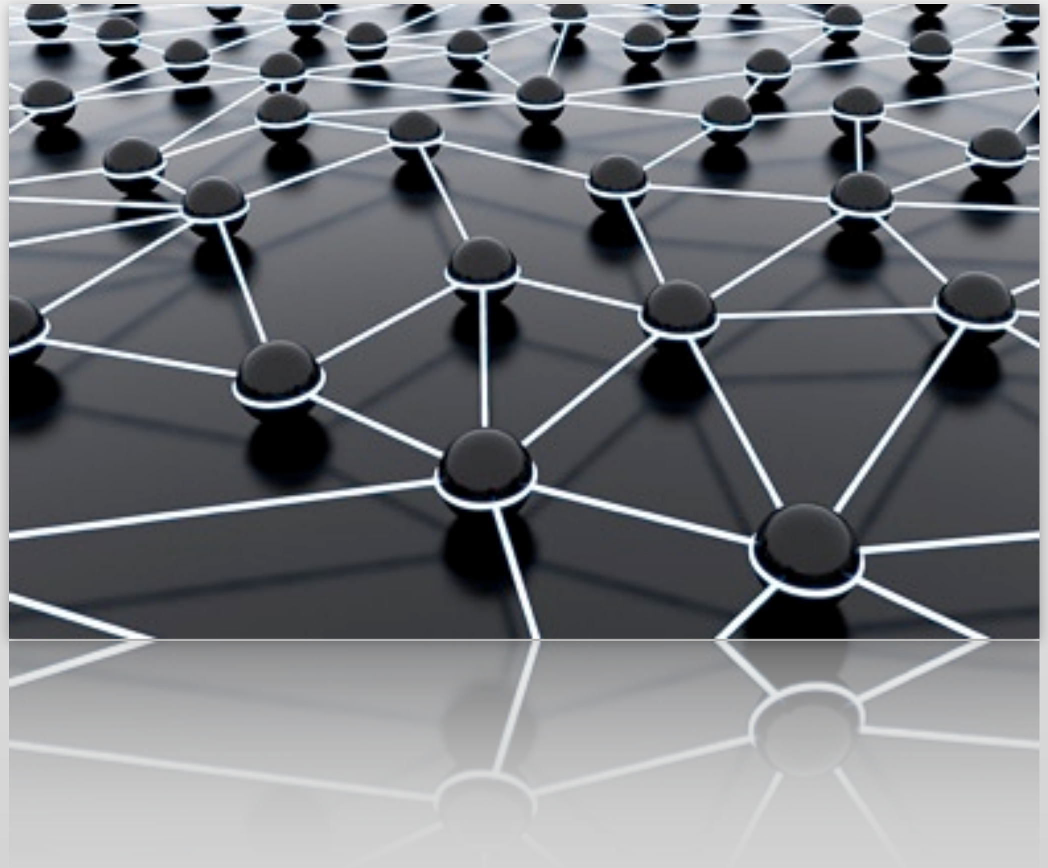
OBJECTIVES

- Two objectives are mainly considered in the paper
- MaxSave: Given `B` and `T⊆V`, find valid vaccination strategy that maximizes number of vertices saved in `T`
  - When `T=V`, MaxSave corresponds to the well known Firefighter problem
- MinBudget: Given `T⊆V`, find valid vaccination strategy that saves all vertices in `T` and minimizes `B`



RELATED WORK

- Problem introduced by B. Hartnell in 1995
- Much work on Firefighter problem has been focussed on special graphs like grids and usually for MaxSave [DH'07], [Fogarty'03],[WM'03]
- Approximation results for trees [HL'00], [LVY'08], [CC'10]



RESULTS

	Spreading	Non-spreading
Max-Save	$(1-1/e)$ approx	$n^{(1-\varepsilon)}$-hard for any $\varepsilon>0$
Min-Budget	$\log(n)$ approx $\Omega(\log n)$-hard	General: $O(\sqrt{n})$ approx For Directed L-layered Graphs: $O(\log L)$ approx (independently [CC'09])

MODEL: SPREADING

OBJECTIVE: MAXSAVE

Given Budget **B**: Maximize nodes saved in **T**

MODEL: SPREADING

OBJECTIVE: MAXSAVE

Given Budget **B**: Maximize nodes saved in **T**

- MaxSave problem for spreading model is NP-complete

MODEL: SPREADING

OBJECTIVE: MAXSAVE

Given Budget **B**: Maximize nodes saved in **T**

- MaxSave problem for spreading model is NP-complete
- Approximation for MaxSave:

MODEL: SPREADING

OBJECTIVE: MAXSAVE

Given Budget **B**: Maximize nodes saved in **T**

- MaxSave problem for spreading model is NP-complete
- Approximation for MaxSave:
  - Set of valid immunizations can be mapped to partition matroid

MODEL: SPREADING

OBJECTIVE: MAXSAVE

Given Budget **B**: Maximize nodes saved in **T**

- MaxSave problem for spreading model is NP-complete
- Approximation for MaxSave:
  - Set of valid immunizations can be mapped to partition matroid
  - Number saved by valid immunizations is a submodular set function

MODEL: SPREADING

OBJECTIVE: MAXSAVE

Given Budget $\mathbf{B}$: Maximize nodes saved in $\mathbf{T}$

- MaxSave problem for spreading model is NP-complete
- Approximation for MaxSave:
  - Set of valid immunizations can be mapped to partition matroid
  - Number saved by valid immunizations is a submodular set function
  - Finding the optimal solution is akin to maximizing a set function over all valid immunizations

MODEL: SPREADING

OBJECTIVE: MAXSAVE

Given Budget **B**: Maximize nodes saved in **T**

- MaxSave problem for spreading model is NP-complete
- Approximation for MaxSave:
  - Set of valid immunizations can be mapped to partition matroid
  - Number saved by valid immunizations is a submodular set function
  - Finding the optimal solution is akin to maximizing a set function over all valid immunizations
  - Can get $(1-1/e)$ approximation by using recent result [CCPV'07]

MODEL: SPREADING

OBJECTIVE: MAXSAVE

Given Budget **B**: Maximize nodes saved in **T**

- MaxSave problem for spreading model is NP-complete
- Approximation for MaxSave:
  - Set of valid immunizations can be mapped to partition matroid
  - Number saved by valid immunizations is a submodular set function
  - Finding the optimal solution is akin to maximizing a set function over all valid immunizations
  - Can get **(1-1/e)** approximation by using recent result [CCPV'07]
  - Also ***Greedy*** 2-approximate solution

MODEL: SPREADING

OBJECTIVE: MINBUDGET

Save all nodes in given set T: Minimize Budget B

- MinBudget on directed graphs is as hard as Set Cover
- This implies inapproximability to the factor of $\log n$
- An iterative greedy algorithm gives $\log n$ factor approximation
- Same result obtained by applying randomized rounding

RESULTS

	Spreading	Non-spreading
Max-Save	$(1-1/e)$ approx	$n^{(1-\varepsilon)}$-hard for any $\varepsilon>0$
Min-Budget	$O(\log n)$ approx $\Omega(\log n)$-hard	General: $O(\sqrt{n})$ approx For Directed L-layered Graphs: $O(\log L)$ approx (independently [CC'10])

RESULTS

	Spreading	Non-spreading
Max-Save	$(1-1/e)$ approx	$n^{(1-\varepsilon)}$-hard for any $\varepsilon>0$
Min-Budget	$O(\log n)$ approx $\Omega(\log n)$-hard	General: $O(\sqrt{n})$ approx For Directed L-layered Graphs: $O(\log L)$ approx (independently [CC'10])

RESULTS

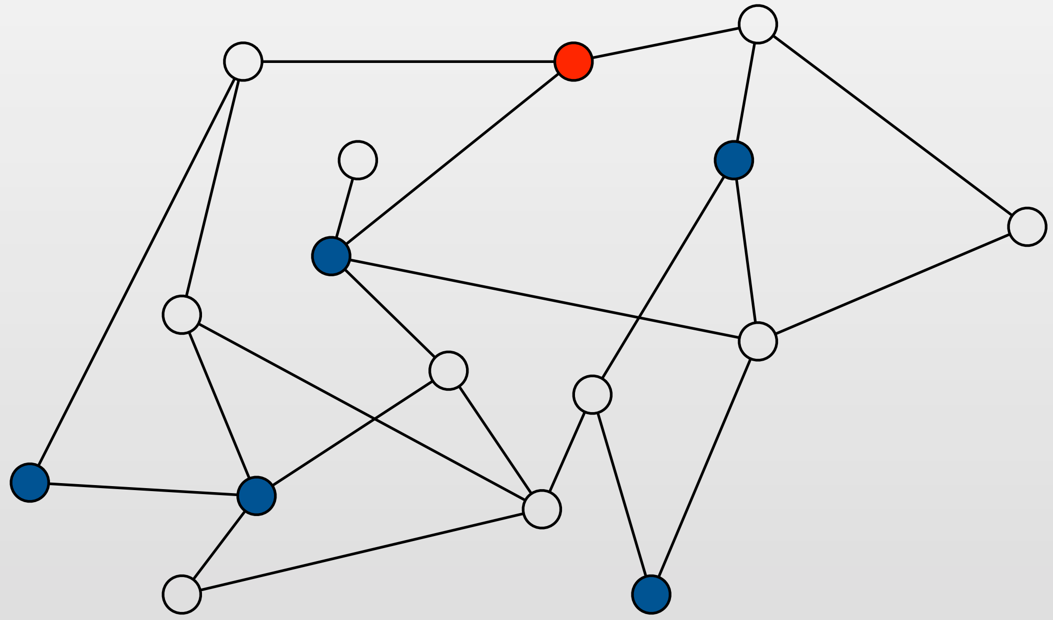
	Spreading	Non-spreading
Max-Save	$(1-1/e)$ approx	$n^{(1-\varepsilon)}$-hard for any $\varepsilon>0$
Min-Budget	$O(\log n)$ approx $\Omega(\log n)$-hard	General: $O(\sqrt{n})$ approx For Directed L-layered Graphs: $O(\log L)$ approx (independently [CC'10])

RESULTS

	Spreading	Non-spreading
Max-Save	$(1-1/e)$ approx	$n^{(1-\varepsilon)}$-hard for any $\varepsilon>0$
Min-Budget	$O(\log n)$ approx $\Omega(\log n)$-hard	General: $O(\sqrt{n})$ approx For Directed L-layered Graphs: $O(\log L)$ approx (independently [CC'10])

MODEL: NON-SPREADING
OBJECTIVE: MINBUDGET

Save all nodes in given set **T**: Minimize Budget **B**

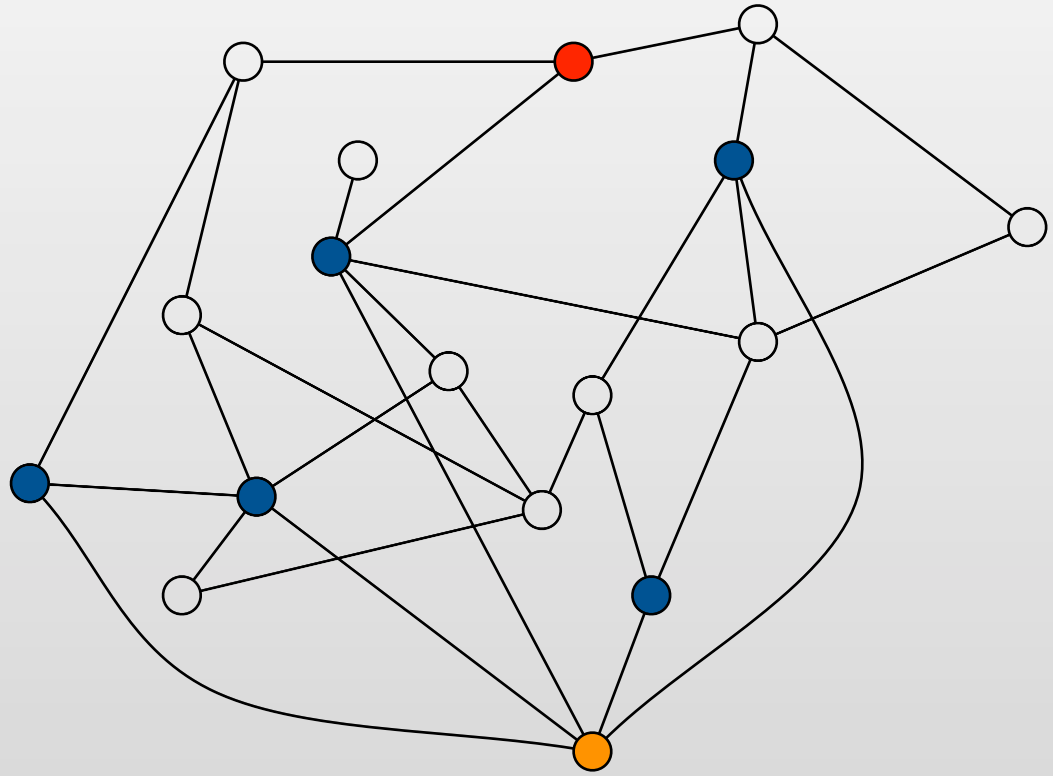


- Simplification: adding $\mathtt{t_{sink}}$ vertex

MODEL: NON-SPREADING
OBJECTIVE: MINBUDGET

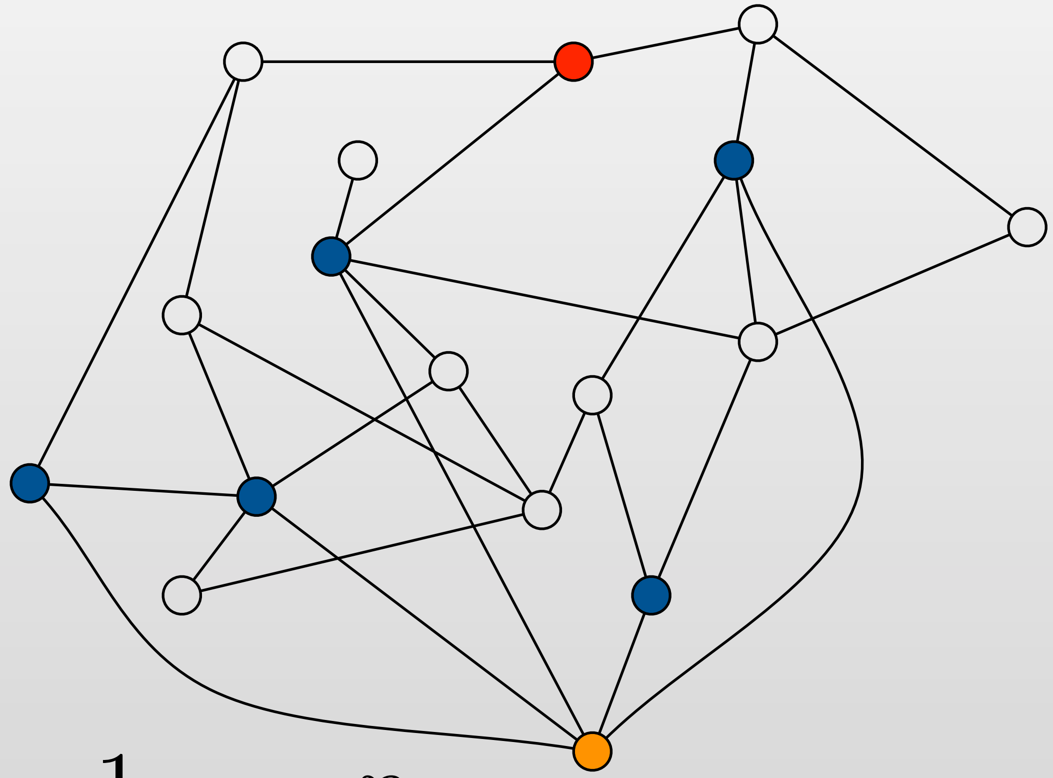
Save all nodes in given set $\mathtt{T}$: Minimize Budget $\mathtt{B}$

- Simplification: adding $\mathtt{t_{sink}}$ vertex



MODEL: NON-SPREADING
OBJECTIVE: MINBUDGET

Save all nodes in given set **T**: Minimize Budget **B**



- Simplification: adding t_{sink} vertex

Integer program:

$x_v^t = 1$ … if vertex **v** is vaccinated at time **t**

0 … if not

Minimize **B**

Subject to:

$$\sum_{v \in V} x_v^t \leq B \qquad \forall t = 1, \ldots, n$$

$$\sum_{i=1}^{k} \sum_{t=1}^{i} x_{v_i}^t \geq 1 \qquad \forall (s, v_1, \cdots, v_k, t) \in P$$

$$x_v^t \geq 0 \qquad \forall v \in V,\ t = 1, 2, \cdots, n$$

MODEL: NON-SPREADING
OBJECTIVE: MINBUDGET

Save all nodes in given set T: Minimize Budget B

General Graphs

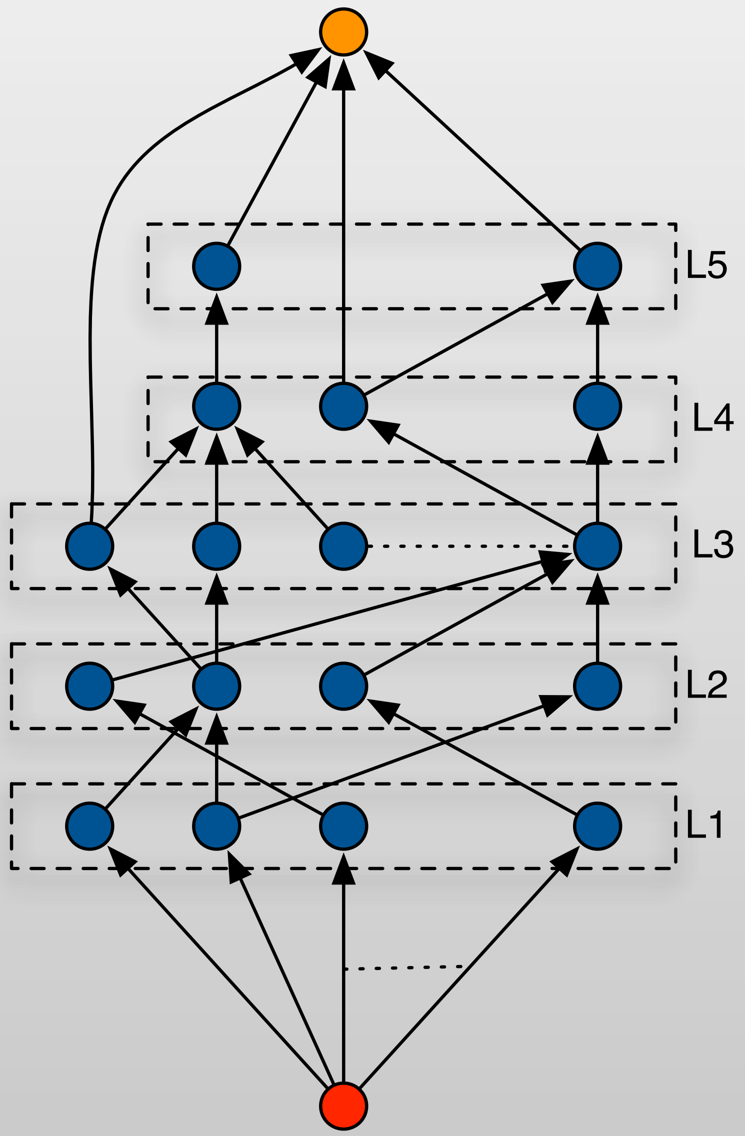
- A $2\sqrt{n}$ approximation
  - Consider the LP relaxation of the integer program
  - Vertex is vaccinated at time i if fractionally cut by amount $1/\sqrt{n}$ till time i
  - In the resulting graph, all s-t paths are at least $\sqrt{n}$ hops long
  - This graph has min-cut of size at most $\sqrt{n}$

MODEL: NON-SPREADING
OBJECTIVE: MINBUDGET

Save all nodes in given set **T**: Minimize Budget **B**

Directed L-layered Graphs

- We give an example of layered graph that has integrality gap of size at least $\mathbf{H}_L$ (where $\mathbf{H}_n = 1+1/2+ \ldots +1/n$)



MODEL: NON-SPREADING
OBJECTIVE: MINBUDGET

Save all nodes in given set **T**: Minimize Budget **B**

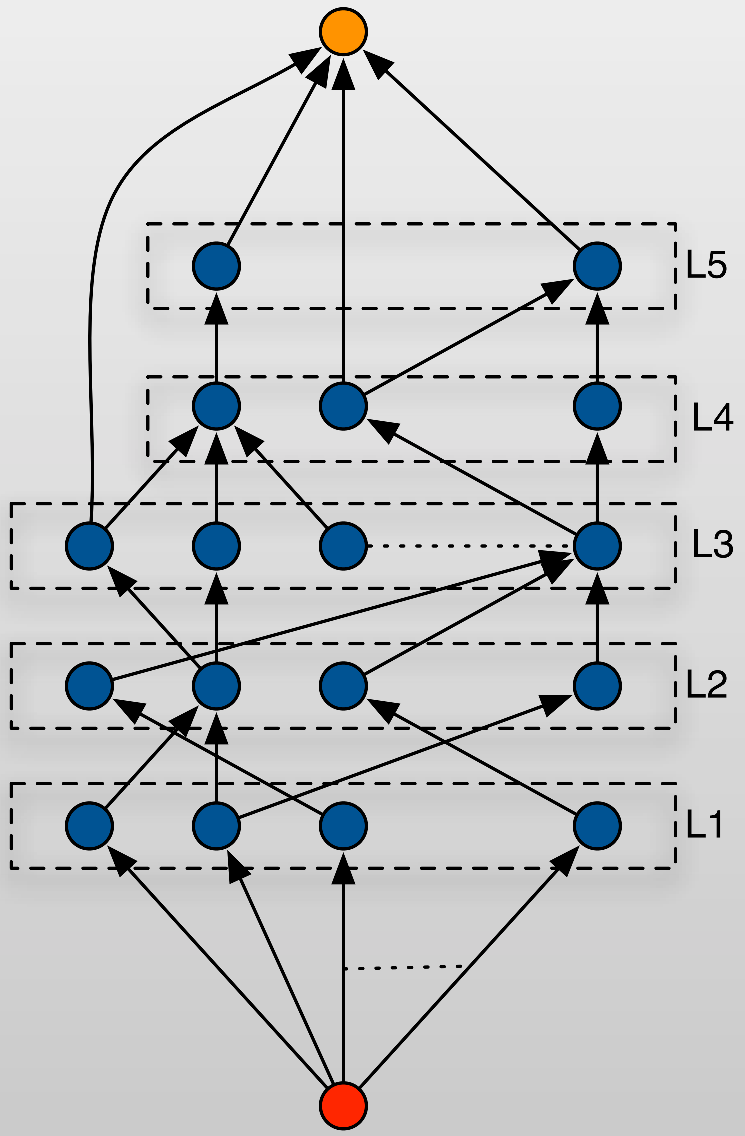
Directed L-layered Graphs

- We give an example of layered graph that has integrality gap of size at least $\mathbf{H}_L$ (where $\mathbf{H}_n = 1+1/2+ \ldots +1/n$)

- Algorithm for $\mathbf{O}(\log L)$ approximation:

  1. Set the capacity of each $v \in L_i$ as $1/iH_L$

  2. Find min-cut with above capacities. Let it be $(N_1 \cup N_2 \cup \ldots \cup N_L)$ with $N_i \subseteq L_i$



MODEL: NON-SPREADING
OBJECTIVE: MINBUDGET

Save all nodes in given set T: Minimize Budget B

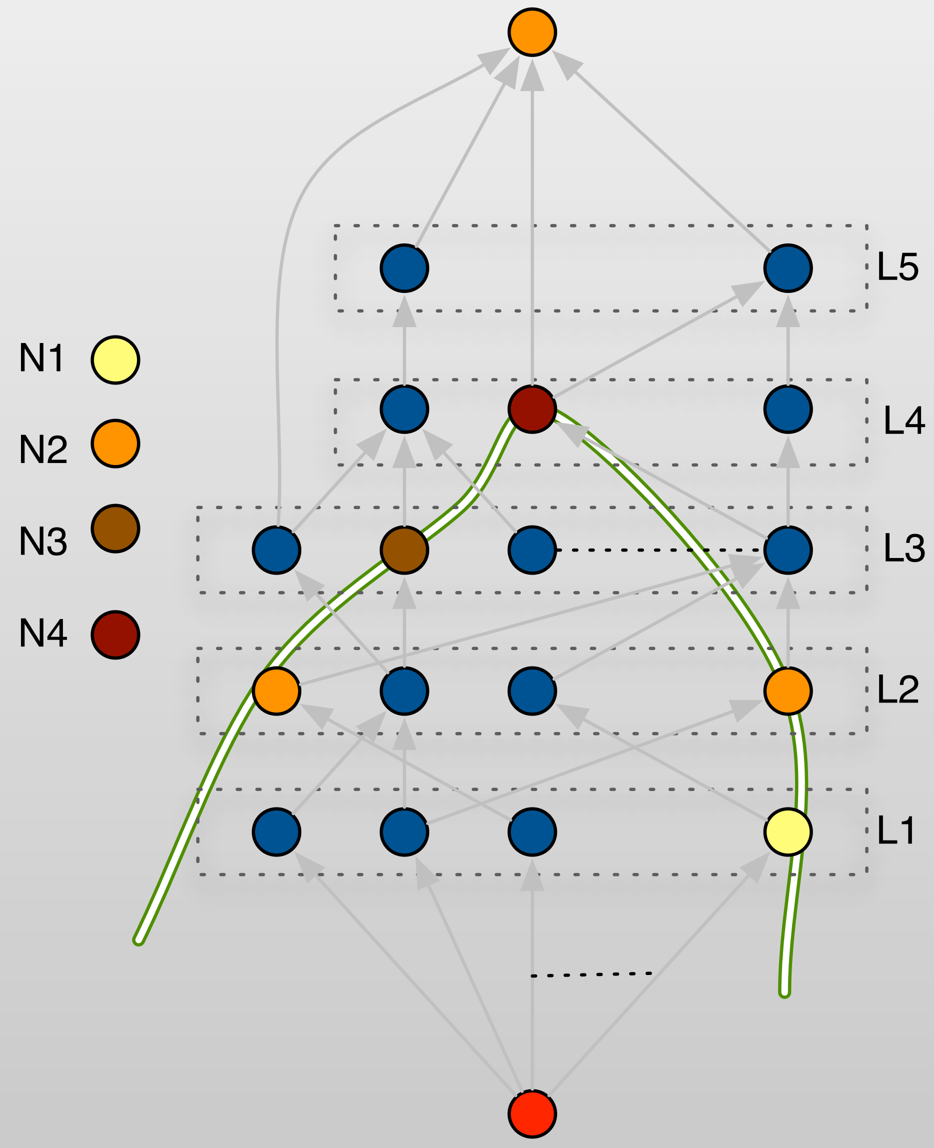
Directed L-layered Graphs

- We give an example of layered graph that has integrality gap of size at least H_L (where $H_n = 1+1/2+ \ldots +1/n$)

- Algorithm for $O(\log L)$ approximation:

    1. Set the capacity of each $v \in L_i$ as $1/iH_L$

    2. Find min-cut with above capacities. Let it be $(N_1 \cup N_2 \cup \ldots \cup N_L)$ with $N_i \subseteq L_i$



MODEL: NON-SPREADING

OBJECTIVE: MINBUDGET

Save all nodes in given set T: Minimize Budget B

… Directed L-layered Graphs

3. Vaccination strategy:

- Day one: $|N_1|$ vertices of N_1, $|N_2|/2$ vertices of N_2 and so on.
- Day two: $|N_2|/2$ vertices of N_2, $|N_3|/3$ vertices of N_3 and so on.
- In general on day i: $|N_j|/j$ vertices of N_j for $i \leq j \leq L$

RESULTS

	Spreading	Non-spreading
Max-Save	$(1-1/e)$ approx	$n^{(1-\varepsilon)}$-hard for any $\varepsilon>0$
Min-Budget	$O(\log n)$ approx $\Omega(\log n)$-hard	General: $O(\sqrt{n})$ approx For Directed L-layered Graphs: $O(\log L)$ approx (independently [CC'10])

Trees

- Max-Save and Min-Budget are NP-Complete even for degree 3
- Spreading and Non-Spreading Models are equivalent
- Thus we have (1-1/e)-approx for Max-Save
- Thus we have (log L)-approx for Min-Budget
- [Chalermsook + Chuzhoy] recently gave $O(\log^* n)$-approx

OPEN PROBLEMS

- Bring down $\sqrt{n}$ approximation for Min-Budget on Non-spreading model
- Rate of spread of vaccination may lie somewhere between 0 and 1
- Probabilistic infection spread
- Threshold based models of infection spread

QUESTIONS?