FINAL: 180 Minutes

Answer **ALL** questions.
OPEN BOOK (notes, assignments, and textbook) and electronic devices allowed.
NO COLLABORATION or Internet use. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.
You **MUST** show **CORRECT** work, **even on multiple choice questions**, to get credit.

GOOD LUCK!

1	2	3	4	5	6	Total
200	**30**	**30**	**30**	**30**	**30**	**350**

1 Circle one answer per question. 10 points for each correct answer.

(1) In how many ways can six indistinguishable balls be distributed into nine distinguishable bins if each bin can hold up to six balls?

$\boxed{\text{A}}$ 54

$\boxed{\text{B}}$ 2000

$\boxed{\text{C}}$ 3003

$\boxed{\text{D}}$ 84

$\boxed{\text{E}}$ None of the above

(2) If n is a positive integer, then how many nonempty bit strings with length not exceeding n consist entirely of ones?

$\boxed{\text{A}}$ n+1

$\boxed{\text{B}}$ n-1

$\boxed{\text{C}}$ n

$\boxed{\text{D}}$ 1

$\boxed{\text{E}}$ None of the above

(3) A box contains a dozen brown socks and a dozen black socks, all unmatched. If a person takes out socks randomly without replacement, how many socks must she take out to be sure that she has at least two socks of the same color?

$\boxed{\text{A}}$ 2

$\boxed{\text{B}}$ 1

$\boxed{\text{C}}$ 4

$\boxed{\text{D}}$ 3

$\boxed{\text{E}}$ None of the above

(4) You are given a deck of 52 cards. You draw from the deck uniformly at random and put the card back. You continue this procedure until you have seen all four aces. What is the expected value of the number of draws you take before seeing all the aces?

$\boxed{\text{A}}$ 100.06

$\boxed{\text{B}}$ 52

$\boxed{\text{C}}$ 61.5

$\boxed{\text{D}}$ 108.33

$\boxed{\text{E}}$ None of the above

(5) At a birthday party, three types of fruit juices are served to the guests. The supply of juice bottles consists of eight orange juices, 10 apple juices, and 12 fruit punches. If six bottles are randomly selected, what is the probability that all of them are the same variety?

A 0.14

B 0.01

C 0.0056

D 0.002

E None of the above

(6) What is the probability that, in a group of n people, at least two were born in the same month of the year? You may assume $0 < n \leq 12$.

A $\frac{11}{12} \cdot \frac{10}{12} \cdots \frac{13-n}{12}$

B $\frac{13-n}{12}$

C $1 - \frac{n}{12}$

D $1 - \frac{11}{12} \cdot \frac{10}{12} \cdots \frac{13-n}{12}$

E None of the above

(7) Which of the following pairs satisfy $f \in o(g)$?

(I) $f(n) = n^6$, $g(n) = 10n^4$

(II) $f(n) = 5n^2 + 1$, $g(n) = n^2 \log n$

(III) $f(n) = n^{\log n}$, $g(n) = (\log n)^n$

(IV) $f(n) = \cos^2(n\pi)$, $g(n) = \sin^2(n\pi)$

A I

B II, IV

C II, III

D I, II, III

E II, III, IV

(8) In a nearby gas station, 40% of the customers use regular gas, 35% use plus gas and 25% use premium gas. Of those customers using regular gas, only 30% fill their tanks; 60% of plus customers fill their tanks, and 50% of premium customers fill their tanks. If the next customer fills their tank, what is the probability that they used premium gas?

A 0.275

B 0.462

C 0.125

D 0.210

E None of the above

(9) What is 7^{21} mod 43?

$\boxed{\text{A}}$ 1

$\boxed{\text{B}}$ 6

$\boxed{\text{C}}$ 11

$\boxed{\text{D}}$ 36

$\boxed{\text{E}}$ 42

(10) If a graph on n vertices is disconnected, what is the largest number of edges it can have?

$\boxed{\text{A}}$ $n-1$

$\boxed{\text{B}}$ n

$\boxed{\text{C}}$ $\binom{n-1}{2}$

$\boxed{\text{D}}$ $\binom{n-1}{2}-1$

$\boxed{\text{E}}$ $\binom{n}{2}-1$

(11) In a clinic, 8% of the patients are infected with a virus. When a blood test is given for this virus, 98% of the infected patients test positive and 3% of the uninfected test positive. What is the probability that a patient who tests negative is infected?

$\boxed{\text{A}}$ 0.740

$\boxed{\text{B}}$ 0.260

$\boxed{\text{C}}$ 0.002

$\boxed{\text{D}}$ 0.998

$\boxed{\text{E}}$ None of the above

(12) What is the expected value of the sum of numbers appearing on two fair dice, given that the sum of these numbers is at least 9?

$\boxed{\text{A}}$ 9

$\boxed{\text{B}}$ 10

$\boxed{\text{C}}$ 12

$\boxed{\text{D}}$ 6

$\boxed{\text{E}}$ 8

(13) Which of the following implies $\mathcal{L} \in \mathcal{P}$?

(I) $\mathcal{L}$ is regular
(II) $\mathcal{L}$ is Turing decidable
(III) $\overline{\mathcal{L}} \in \mathcal{P}$

$\boxed{\text{A}}$ I

$\boxed{\text{B}}$ II

C III

D I, II

E I, III

(14) $\mathcal{L} = \{x \mid x \text{ is a binary expansion of } \sqrt{2}\}$ can be solved by:

(I) DFA
(II) CFG
(III) Turing Machine

A all

B I,III

C II,III

D III

E none

(15) Evaluate the sum $T(n) = \sum_{i=1}^{n} \left[\left(\frac{1}{2}\right)^i + \sum_{j=1}^{n} j \right]$.

A $\frac{1}{2}\left(n^3 + n^2 + 4 - \frac{1}{2^{n-1}}\right)$

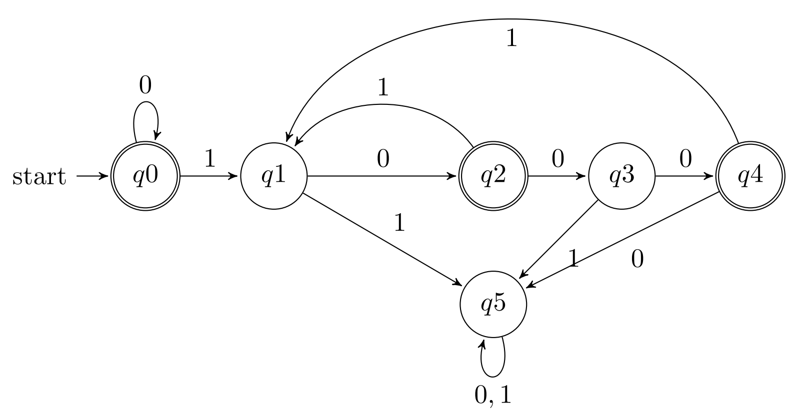
B $\frac{1}{2}\left(n^3 + n^2 + 4 - \frac{1}{2^{n}}\right)$

C $\frac{1}{2}\left(n^3 + n^2 + 2 - \frac{1}{2^{n}}\right)$

D $\frac{1}{2}\left(n^3 + n^2 + 2 - \frac{1}{2^{n-1}}\right)$

E $\frac{1}{2}\left(n^3 + 2n^2 + n + 4 - \frac{1}{2^{n-1}}\right)$

(16) What language does the following DFA accept?



A $\{0\}^* \bullet 1 \bullet \{01, 0001\}^\star$

B $\{0\}^* \bullet \{100\}^*$

C $\{0, 1\}^* \bullet \{1000\}^*$

D $\{0,1\}^* \bullet \{100\}^*$

E $\{0\}^* \bullet \{100, 1000\}^*$

(17) If you have seven colors to choose from, how many different colorings of K_5 can you provide?

A 21

B 35

C 120

D 2520

E 5040

(18) Which is logically equivalent to $\neg((p \wedge q) \vee r)$?

A $(p \vee \neg r) \vee (q \wedge \neg r)$

B $(\neg p \vee r) \wedge (\neg q \wedge r)$

C $(\neg p \wedge r) \vee (q \wedge \neg r)$

D $(\neg p \vee \neg r) \wedge (\neg q \vee \neg r)$

E $(\neg p \wedge \neg r) \vee (\neg q \wedge \neg r)$

(19) What is the contrapositive of "If a millenial didn't inherit money or win the lottery, then they do not own a home"?

A If a millenial inherits money and wins the lottery, then they own a home.

B If a millenial owns a home, then they inherited money or won the lottery.

C If a millenial inherits money or wins the lottery, then they own a home.

D If a millenial owns a home, then they inherited money and won the lottery.

E If a millenial does not own a home, then they did not inherit money or win the lottery.

(20) Which string below is not in the language of the CFG

1: $S \rightarrow 0T1|T1$
2: $T \rightarrow 00T1|\varepsilon$

A 00000111

B 1

C 00111

D 0011

E 0000111

2 **You play a game in which you pay one dollar to blindly draw three balls from a box, uniformly at random without replacement. The box contains ten balls; four of these balls are golden. You get back your original one-dollar stake if you draw exactly two golden balls, while you win ten dollars plus your original one-dollar stake if you draw three golden balls. Otherwise, you get nothing. What is your expected loss after playing this game?**

3 **Prove that, for any random variable X that takes on values in the finite set $\{a_1, \ldots, a_n\}$, there is at least one a_i in this set that satisfies**

$$a_i \geq \mathbb{E}X.$$

Clearly identify the type of proof you used.

4 Write pseudocode (at the level of the examples given in class) for solving the problem $\mathcal{L} = \{0^{\bullet n} 1^{\bullet (n+2)} \mid n > 0\}$ using a Turing Machine.

5 **Let** $T_1 = 1$**,** $T_2 = 2$ **and** $T_n = (T_{n-1} + T_{n-2})/2$ **when** $n \geq 3$**. Find a closed form expression for** T_n **that applies when** $n \geq 3$**.**

6 Prove that $(3^{77} - 1)/2$ is odd.

SCRATCH

SCRATCH