CSCI 6220/4030: Homework 1

Assigned Thursday September 14 2017. Due at beginning of class Thursday September 28 2017.

Remember to typeset your submission, and label it with your name. Please start early so you have ample time to see me during office hours. Provide mathematically convincing arguments for 4 of the following problems (for your own benefit, choose a varied selection). Ask me if you are unclear whether your arguments are acceptable.

1. Suppose we roll n standard 6-sided dice. What is the probability that their sum is divisible by 6, assuming the rolls are independent?

2. We roll a standard fair die over and over. What is the expected number of rolls until the first pair of consecutive sixes appears?

3. Generalize the notion of a cut set to an r-way cut as a set of edges whose removal breaks the graph into r or more connected components. Explain how randContract can be used to find minimum r-way cut-sets, and bound the probability that it succeeds in one iteration.

4. Suppose we flip a coin n times to obtain a sequence of flips $X_1, \ldots, X_n$. A streak of flips is a consecutive subsequence of flips that are all the same. For example, if X_3, X_4, and X_5 are all heads, there is a streak of length 3 starting at the third flip. (NB: if X_6 is also heads, then there is also a streak of length 4 starting at the third flip.) Let n be a power of 2. Show that the expected number of streaks of length $\log_2 n + 1$ is $1 - o(1)$.

5. Let $A[1], \ldots, A[n]$ be an array formed from a uniform random permutation of the numbers 1 through n. One pass of the bubblesort algorithm moves through the array from $i = 1$ to $i = n - 1$ and swaps $A[i]$ and $A[i+1]$ if $A[i] > A[i+1]$. As a function of n, what is the expected number of swaps performed in one pass of bubblesort?

6. Let Random(i, n) return a sample of a uniform random choice of an integer in the interval $[i, n]$. Show that, given an array $A[1], \ldots, A[n]$, the following algorithm returns an A whose entries have been uniformly randomly permuted:

   for $i \leftarrow 1, n$ **do**
       $r \leftarrow$ Random(i, n)
       Exchange $A[i]$ and $A[r]$

7. Linear insertion sort can sort an array of numbers in place. The first and second numbers are compared; if they are out of order, they are swapped so that they are in sorted order. The third number is then placed in the appropriate place in the sorted order. It is first compared with the second; it if is not in the proper order, it is swapped and compared with the first. Iteratively, the kth number is handled by swapping it downwards until the first k numbers are in sorted order. Determine the expected number of swaps that need to be made with a linear insertion sort when the input is a random permutation of n distinct numbers.

8. A blood test is being performed on n individuals. Each person can be tested separately, but this is expensive. Pooling can decrease the cost. The blood samples of k people can be pooled and analyzed together. If the test is negative, this one test suffices for the group of k individuals. If the test is positive, then each of the k persons must be tested separately and thus $k + 1$ total tests are required for the k people.

   Suppose that we create n/k disjoint groups of k people (where k divides n) and use the pooling method. Assume that each person has a positive result on the test independently with probability p.

   (a) What is the probability that the test for a pooled sample of k people will be positive?
   (b) What is the expected number of tests necessary?

(c) Describe how to find the best value of k.

(d) Give an inequality that show for what values of p pooling is better than just testing every individual.

9. MR95, Problem 1.6
10. MR95, Problem 3.1
11. MR95, Exercise 3.1
12. MR95, Problem 3.5
13. MR95, Problem 3.12