CSCI 6220/4030: Homework 5

Assigned Thursday November 16 2017. Due at beginning of class Thursday November 30 2017.

Remember to typeset your submission, and label it with your name. Please start early so you have ample time to see me during office hours. Provide mathematically convincing arguments for the following problems. Ask me if you are unclear whether your arguments are acceptable.

1. Two rooted trees T_1 and T_2 are said to be isomorphic if there exists a one-to-one mapping f from the vertices of T_1 to those of T_2 satisfying the following condition: for each internal vertex v of T_1 with the children $v_1, \ldots, v_k$, the vertex $f(v)$ has as children exactly the vertices $f(v_1), \ldots, f(v_k)$. Observe that no ordering is assumed on the children of any internal vertex. Devise an efficient randomized algorithm for testing the isomorphism of rooted trees using the Schwarzt-Zippel theorm, and analyze its cost and probability of success. Hint: associate a polynomial P_v with each vertex v in a tree T. The polynomials are defined recursively, with the base case being that the leaf vertices all have $P = x_0$. An internal vertex v of height h with the children $v_1, \ldots, v_k$ has its polynomial defined to be $(x_h - P_{v_1}) \cdots (x_h - P_{v_k})$. Note that there is one indeterminate x_h associated with each level h in the tree.

2. Consider two computers, each containing n bit strings of length n. It can be shown that any deterministic algorithm for determining whether these sets have a non-empty intersection requires $O(n^2)$ bits to be communicated between the computers. Design a Las Vegas algorithm for answering this problem that communicates $O(n \log n)$ bits in expectation *when the sets do not intersect*.

3. Gas molecules move about randomly in a box that is divided into two halves symmetrically by a partition; there is a hole in the partition. Suppose there are n molecules in the box. Molecular motion can be modeled by keeping the molecules in their positions with probability $1/2$, and with probability $1/2$ choosing a number between 1 and n at random and moving the corrresponding molecule to the other side of the partition.

   (a) Show that the number of molecules on one side of the partition evolves as a Markov Chain. What are the states, and the transition probabilities?

   (b) Argue that the chain is ergodic, and find its stationary distribution

   (c) Find the lowest upper-bound on $t_{\text{mix}}(\varepsilon)$ (starting from an arbitrary distribution) that you can. Hint: relate this problem to the random walk on the hypercube.

4. Given a deck of n cards in an arbitrary starting order, consider the following shuffling algorithm: at each step choose one of the cards uniformly at random (including the top card) and move it to the top of the deck. Identify the cards with the numbers $1, \ldots, n$ and argue that the invariant distribution of this Markov Chain is the uniform distribution over S_n, and that $t_{\text{mix}}(\varepsilon) \leq n \ln n + n \ln(1/\varepsilon)$. Hint: to construct the coupling, when you pick a card to move to the top of the X process, pick the same card to move to the top of the Y process.