1. Let $X$ be a Binomial($n, p$) random variable.
   
   (i) Determine the MGF of $X$.
   
   (ii) Use the MGF of $X$ to determine a subgaussian tail bound for $X$.

2. Let $X$ be a Poisson($\lambda$) random variable and $\varepsilon > 0$.
   
   (i) Compute the MGF of $X$.
   
   (ii) Compute the mean, $\mu$, and variance, $\text{Var}(X)$, of $X$.
   
   (iii) Use Chebyshev’s inequality to derive a bound on the probability that $X > (1 + \varepsilon)\mathbb{E}X$.
   
   (iv) Use the Chernoff technique to argue that
   \[ P(X \geq (1 + \varepsilon)\mathbb{E}X) \leq e^{-h(\varepsilon)\text{Var}(X)}, \]
   where $h(\varepsilon) = (1 + \varepsilon) \ln(1 + \varepsilon) - \varepsilon$.
   
   (v) Compare the usefulness of the two bounds.

3. Consider a collection $X_1, \ldots, X_n$ of $n$ independent geometrically distributed random variables with mean 2. Let $X = \sum_{i=1}^{n} X_i$ and $\delta > 0$.
   
   (i) Derive a bound on $P(X \geq (1 + \delta)(2n))$ by relating this probability to the behavior of a sum of $(1 + \delta)2n$ Bernoulli random variables and applying a Chernoff bound.
   
   (ii) Directly derive a Chernoff bound on $P(X \geq (1 + \delta)(2n))$ using the moment generating function for geometric random variables. The form of the bound should be simple.
   
   (iii) Which bound is better in your opinion, and why?

4. [required only for CSCI6220] Consider $n$ balls thrown randomly into $n$ bins. Let $X_i = 1$ if the $i$th bin is empty and 0 otherwise. Let $X = \sum_{i=1}^{n} X_i$. Note that $X$ is not the sum of independent random variables, so we cannot use a Chernoff bound directly.
   
   Instead, we will show that the MGF of $X$ is smaller than the MGF of a sum of independent r.v.s and obtain a Chernoff bound in terms of the latter. To do so, let $Y_i$ for $i = 1, \ldots, n$ be independent Bernoulli random variables that are 1 with probability $p = (1 - 1/n)^n$. Let $Y = \sum_{i=1}^{n} Y_i$.
   
   (i) What is the probability that $X_i = 1$?
   
   (ii) Give an intuitive argument for why we should expect $Y$ to be greater than $X$.
   
   (iii) Show that $\mathbb{E}[X_1 X_2 \cdots X_k] \leq \mathbb{E}[Y_1 Y_2 \cdots Y_k]$ for any $k \geq 1$.
   
   (iv) Show that $M_X(\lambda) = \mathbb{E}[e^{\lambda X}] \leq \mathbb{E}[e^{\lambda Y}] = M_Y(\lambda)$ for all $\lambda \geq 0$. (Hint: use the expansion for $e^x$ and compare $\mathbb{E}[X^k]$ to $\mathbb{E}[Y^k]$.)
   
   (v) Derive a Chernoff bound for $P(X \geq (1 + \delta)\mathbb{E}[X])$. 