1. $\sqrt{3}$ is what kind of number?

$\boxed{\text{A}}$ A natural number.

$\boxed{\text{B}}$ A rational number.

$\boxed{\text{C}}$ An irrational number.

$\boxed{\text{D}}$ An integer.

$\boxed{\text{E}}$ None of the above.

2. Find the correct expression for the recurrence given by $A_0 = 1$ and $A_n = 3(A_{n-1} + 1) - 1$ when $n \geq 1$.

$\boxed{\text{A}}$ $A_n = 2 \cdot 3^n - 1$

$\boxed{\text{B}}$ $A_n = 3 \cdot 2^n - 1$

$\boxed{\text{C}}$ $A_n = 5 \cdot 3^n - 4$

$\boxed{\text{D}}$ $A_n = 3 \cdot 4^n - 2$

$\boxed{\text{E}}$ None of the above

Base case is true $(n=0)$

Induction step $A_{n-1} = 2 \cdot 3^{n-1} - 1$

so $A_n = 3(2 \cdot 3^{n-1} - 1 + 1) - 1$

$= 2 \cdot 3^n - 1$

3. Which of the following is equivalent to the proposition $\forall x : (\neg \exists y : R(x,y))$?

equiv to $\forall y : \neg R(x,y)$

$\boxed{\text{A}}$ $\exists x : \forall y : \neg R(x,y)$

$\boxed{\text{B}}$ $\forall x : \forall y : \neg R(x,y)$

$\boxed{\text{C}}$ $\forall x : \exists y : R(x,y)$

$\boxed{\text{D}}$ $\exists x : \forall y : \neg R(x,y)$

$\boxed{\text{E}}$ None of the above.

4. An integer $n \in \mathbb{Z}$ has an odd square, that is n^2 is odd. Which claim is true?

$\boxed{\text{A}}$ n is positive.

$\boxed{\text{B}}$ n^2 is divisible by 3.

$\boxed{\text{C}}$ n is odd. ← can show "n^2 is odd implies n is odd" via contraposition

$\boxed{\text{D}}$ n is divisible by 3.

$\boxed{\text{E}}$ None of the above claims are true.

5. S is recursively defined as follows: $1 \in S$, $2 \in S$, and if $a, b \in S$, then $ab + 1 \in S$. Which of the following is *not* true about S?

$S = \{1, 2, \underset{1\cdot 2+1}{3}, \underset{1\cdot 3+1}{4}, \ldots\} = \mathbb{N}$

$\boxed{\text{A}}$ S contains all the primes.

$\boxed{\text{B}}$ $51 \in S$.

$\boxed{\text{C}}$ All powers of 2 are in S.

$\boxed{\text{D}}$ Given an element $x \in S$ that is not 1 or 2, the pair (a,b) that satisfies $ab + 1 = x$ is unique.

$\boxed{\text{E}}$ All of the above are true.

e.g. $9 = 2 \cdot 4 + 1$
$= 8 \cdot 1 + 1$

6. Which of the following captures the proposition "For p to be true, it is sufficient that q be true"?

[A] $p \to q$

[B] $q \to p$

[C] $p \leftrightarrow q$

[D] $\neg q \to \neg p$

[E] None of the above.

7. All that we know of P is that $P(1), P(2), P(3)$ are true and $P(n) \to P(3n)$. We can conclude that P is true for which of the following values of n?

[A] 12

[B] 51

[C] 162

[D] 300

[E] All of the above.

$P(n)$ is true if $n = 3^{\ell}$ for some ℓ
or $n = 2 \cdot 3^{\ell}$

8. Which of the following is *not* equivalent to $p \leftrightarrow q$?

[A] $(\neg p \to \neg q) \wedge (\neg q \to \neg p)$

[B] $(p \to q) \wedge (\neg q \to \neg p)$

[C] $(\neg p \vee q) \wedge (\neg q \vee p)$

[D] $(p \to q) \wedge (\neg p \to \neg q)$

[E] All of the above are equivalent.

$\neg q \to \neg p$ is equivalent to $p \to q$

9. Which of the following is the negation of "There is a student who got As on all the assignments and attended all lectures, but did not pass FOCS"? Let $A(x)$ denote "x got As on all assignments", $L(x)$ denote "x attended all lectures", and $P(x)$ denote "x passed FOCS".

[A] $\forall x : A(x) \wedge L(x) \wedge P(x)$

[B] $\exists x : A(x) \wedge L(x) \to P(x)$

[C] $\forall x : P(x) \to A(x) \wedge L(x)$

[D] $\forall x : \neg(A(x) \wedge L(x)) \vee P(x)$

[E] None of the above.

original: $\exists x : A(x) \wedge L(x) \wedge \neg P(x)$

negation: $\forall x : \neg(A(x) \wedge L(x)) \vee P(x)$

10. Which proof technique is most appropriate for showing that the product of any two consecutive integers is even?

[A] Direct.

[B] Leaping Induction.

[C] Contrapositive.

[D] Contradiction.

[E] None of the above.

Exactly one of k and $k+1$ is even,
so the product of the two is even

11. Which proof technique is most appropriate for showing that $p_k \le 2^{2^k}$, where p_k is the kth prime?

- A Direct.
- B Contraposition.
- C Strong Induction.
- D Contradiction.
- E None of the above.

By strong induction: Base $p_1 = 2 \le 2^{2^1}$

Inductive step Assume $p_1, \dots, p_{k-1}$ satisfy, then

$$p_k \le p_1 p_2 \cdots p_{k-1} + 1 \le 2^{2^1 + \cdots + 2^{k-1}} + 1$$
$$\le 2^{2^k - 2} + 1 \le 2^{2^k}$$

12. Consider the recursively defined function $f(n) = f(n/2)$ when $n \in \mathbb{N}$ is even and larger than 1, and $f(n) = f(n-1) + 1$ when $n \in \mathbb{N}$ is odd and larger than 3. How many base cases are needed so this function is well-defined on $\mathbb{N}$?

- A It is already well-defined.
- B 1
- C 2
- D 3
- E None of the above.

$\boxed{f(1)} \to f(2)$ $\quad \boxed{f(3)}$ $\quad f(4) \to f(5)$ $\quad f(6)$

13. What is the difference between using Induction versus Strong Induction to prove $P(n)$ for $n \ge 1$?

- A The base cases are different.
- B Induction is usually easier than Strong Induction.
- C In Induction you prove $P(n+1)$. In Strong Induction you prove $P(n+2)$.
- D In Induction you assume $P(n)$. In Strong Induction you assume $P(1) \wedge P(2) \wedge \cdots \wedge P(n)$.
- E There is no difference between the two methods.

14. Which would be the worst choice of proof technique for establishing $n^8 \le 2^n$ when $n \ge 80$?

- A Leaping Induction.
- B Strong Induction.
- C Weak Induction.
- D Direct.
- E All of the above are equally suitable methods.

} this is an induction appropriate problem

15. Which proof technique should be used to show that there are no rational solutions to $x^2 - 4x + 1 = 0$?

- A Direct.
- B Contrapositive.
- C Contradiction.
- D Induction.
- E None of the above.

This is equivalent to saying the numbers $2 \pm \sqrt{3}$ are irrational. This true because assume the are rational, this would mean $\sqrt{3}$ is rational, which is a contradiction.

16. What are the first four terms A_0, A_1, A_2, A_3 in the recurrence $A_n = \begin{cases} 1 & n = 0; \\ 3A_{n-1} + 2 & n \geq 1. \end{cases}$

A $1, 5, 17, 53$

B $1, 5, 8, 11$

C $1, 3, 6, 9$

D $1, 3, 8, 12$

E None of the above

TYPO on B

17. Let $A = \{7k \mid k \in \mathbb{N}\}$ and $B = \{3k \mid k \in \mathbb{N}\}$. Which statement is true?

A $A \cap B = \emptyset$

B $A \cap B$ has more than one element

C $A \subseteq B$

D $B \subseteq A$

E A and B contain only odd numbers.

$A \cap B = \{x \mid 21 \text{ divides } x\}$

18. How many lines are in the truth table for the proposition $p \to q \vee r$?

A 2

B 6

C 8

D 16

E None of the above

$8 = 2^3$

19. Which is the appropriate proof technique for the claim: n^7 is odd $\to$ n is odd ?

A Direct.

B Contrapositive.

C Contradiction.

D Induction.

E None of the above.

n is even $\Rightarrow n = 2k$ for $k \in \mathbb{Z}$
$\Rightarrow n^7 = 2(2^6k^7)$
$\Rightarrow n^7$ is even

20. For which of the domains $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ is the following statement true: $\forall x : (\exists y : x^2 > y)$?

A $\mathbb{N}$

B $\mathbb{N}$ and $\mathbb{Z}$

C $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$

D $\mathbb{Q}$ and $\mathbb{R}$

E None of the above are correct.

not true for $\mathbb{N}$: consider $x = 1$
true for $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ as x in any of these domains satisfy $x^2 \geq 0 > -1$, so take $y = -1$