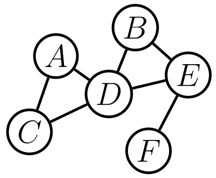
Foundations of Computer Science
Lecture 3

Making Precise Statements

Propositions
Compound Propositions and Truth Tables
Predicates and Quantifiers

Last Time

1. Sets, $\{3, 5, 11\}$
2. Sequences, 100111001
3. Graphs,



4. Examples of basic proofs.
   - In 4 rounds of group dating, no one meets more than 12 people.
   - x^2 is even "is the same as" x is even.
   - In *any* group of 6 people there is an orgy of 3 mutual friends or a war of 3 mutual enemies.
   - **Axiom:** The Well Ordering Principle
   - $\sqrt{2}$ is not rational.

Today: Making Precise Statements

1. Making a precise statement: the proposition

2. Complicated precise statements: the compound proposition
   - Truth tables

3. Claims about many things
   - Predicates
   - Quantifiers
   - Proofs with quantifiers

Statements can be Ambiguous

1. 2+2=4. T
2. 2+2=5. F
3. You may have cake <u>OR</u> ice-cream. (Can you have both?)
4. <u>IF</u> pigs can fly <u>THEN</u> you get an A. (Pigs can't fly. So, can you get an A?)
5. <u>EVERY</u> person has <u>A</u> soul mate.
   - 5(a) There is a single soul mate that <u>EVERY</u> person shares.
   - 5(b) <u>EVERY</u> person has their own special soul mate.

Why is ambiguity bad? **Proof!**

We asked questions of our friends to prove 5(b).

A says Sue's their soul mate;
B says Joe's their soul mate;
C says Sue's their soul mate;
D's soul mate is a red Porshe;
E says Sue's their soul mate;
F says Sam's their soul mate.

Pop Quiz How to prove 5(a)?

Propositions are T or F

We use the letters $p, q, r, s, \ldots$ to represent propositions.

p: Porky the pig can fly.	F
q: You got an A.	T?
r: Kilam is an American.	T?
s: 4^2 is even.	T

To get complex statements, combine basic propositions using logical connectors.

Compound Propositions

p: Porky the pig can fly. F

q: You got an A. T?

r: Kilam is an American. T?

s: 4^2 is even. T

Connector	Symbol	An example in words
NOT	$\neg p$	IT IS NOT THE CASE THAT (Porky the pig can fly)
AND	$p \wedge q$	(Porky the pig can fly) AND (You got an A)
OR	$p \vee q$	(Porky the pig can fly) OR (You got an A)
IF... THEN...	$p \to q$	IF (Porky the pig can fly) THEN (You got an A)

Negation (NOT), $\neg p$

The negation $\neg p$ is T when p is F, and the negation $\neg p$ is F when p is T.

"Porky the pig can fly" is F

So,

IT IS NOT THE CASE THAT (Porky the pig can fly) is T

Conjunction (AND), $p \wedge q$

Both p and q must be T for $p \wedge q$ to be T; otherwise $p \wedge q$ is F.

"Porky the pig can fly" is F

We don't know whether "You got an A".

It does not matter.

(Porky the pig can fly) $\wedge$ (You got an A) is F

Disjunction (OR), $p \vee q$

Both p and q must be F for $p \vee q$ to be F; otherwise $p \vee q$ is T.

"Porky the pig can fly" is F

We don't know whether "You got an A".

Now it matters.

(Porky the pig can fly) $\vee$ (You got an A) is T or F

(Depends on whether you got an A.)

Pop Quiz: "You can have cake" OR "You can have ice-cream." Can you have both?

Truth Tables

p	q	$\neg p$	$p \wedge q$	$p \vee q$
F	F	T	F	F
F	T	T	F	T
T	F	F	F	T
T	T	F	T	T

The truth table defines the "meaning" of these logical connectors.

Implication (IF... THEN...), $p \to q$

<u>IF</u> "Porky the pig can fly" <u>THEN</u> "You got an A." (T/F?)

Suppose T. Since pigs can't fly, does it mean you can't get an A?

<u>IF</u> "n^2 is even", <u>THEN</u> "n is even." (T)

Suppose n^2 is even. Can we conclude $n \neq 5$?

<u>IF</u> "it rained last night" <u>THEN</u> "the grass is wet." (T)

$$\begin{aligned} p &: \text{ it rained last night} \\ q &: \text{ the grass is wet} \end{aligned}$$

$$p \to q$$

What does it *mean* for this common-sense implication to be true?
What can you conclude? Did it rain last night? Is the grass wet?

Adding New Information to a True Implication: p is T

IF "it rained last night" THEN "the grass is wet."

$$\begin{aligned} p &: \text{ it rained last night} \\ q &: \text{ the grass is wet} \end{aligned}$$

$$p \rightarrow q$$

Weather report in morning paper: rain last night. ← new information

IF (it rained last night) THEN (the grass is wet)	T	$p \rightarrow q$	T
It rained last night (from the weather report)	T	p	T
Is the grass wet?	**YES!**	$\therefore\ q$	T

For a **true** implication $p \rightarrow q$, when p is T, you can conclude q is T.

Adding New Information to a True Implication: q is T

IF "it rained last night" THEN "the grass is wet."

$$p : \text{it rained last night}$$
$$q : \text{the grass is wet}$$

$$p \to q$$

While picking up the morning paper, you see wet grass. ← new information

IF (it rained last night) THEN (the grass is wet)	T	$p \to q$	T
The grass is wet (from walking outside)	T	q	T
Did it rain last night?	☹	$\therefore p$	T or F

For a **true** implication $p \to q$, when q is T, you **cannot** conclude p is T.

Adding New Information to a True Implication: p is F

IF "it rained last night" THEN "the grass is wet."

$$p : \text{it rained last night}$$
$$q : \text{the grass is wet}$$

$$p \rightarrow q$$

Weather report in morning paper: no rain last night. ← new information

IF (it rained last night) THEN (the grass is wet)	T	$p \rightarrow q$	T
It rained last night (from the weather report)	F	p	F
Is the grass wet?		$\therefore\ q$	T or F

For a **true** implication $p \rightarrow q$, when p is F, you **cannot** conclude q is F.

Adding New Information to a True Implication: q is F

IF "it rained last night" THEN "the grass is wet."

$$p : \text{it rained last night}$$
$$q : \text{the grass is wet}$$

$$p \rightarrow q$$

While picking up the paper, you see dry grass. ← new information

IF (it rained last night) THEN (the grass is wet)	T		$p \rightarrow q$	T
It grass is wet (from walking outside)	F		q	F
Did it rain last night?	🙂	$\therefore$	p	F

For a **true** implication $p \rightarrow q$, when q is F, you can conclude p is F.

Implication: Inferences When New Information Comes

For a **true** implication $p \to q$:

When p is T, you can conclude that q is T.

When q is T, you **cannot** conclude p is T.

When p is F, you **cannot** conclude q is F.

When q is F, you can conclude p is F.

$$\text{IF } \underbrace{\text{(Porky the pig can fly)}}_{\text{F}} \text{ THEN } \underbrace{\text{(You got an } A\text{)}}_{\text{can be T or F (phew)}}$$

Falsifying "IF (it rained last night) THEN (the grass is wet)"

- You are a scientist collecting data to *verify* that the implication is valid (true).
- **One night it rained. That morning the grass was dry.** ← new information
- What do you think about the implication now?

This is a *falsifying scenario*.

IF (it rains) THEN (the grass is wet) ← not T

$p \to q$ is F *only* when p is T and q is F. In all other cases $p \to q$ is T.

Implication is *Extremely* Important, $p \to q$

All these are $p \to q$ ($p =$ "it rained last night" and $q =$ "the grass is wet"):

If it rained last night then the grass is wet.	IF p THEN q
It rained last night implies the grass is wet.	p IMPLIES q
It rained last night only if the grass is wet.	p ONLY IF q
The grass is wet if it rained last night.	q IF p
The grass is wet whenever it rains.	q WHENEVER p

Truth Tables:

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$\boldsymbol{p \to q}$
F	F	T	F	F	T
F	T	T	F	T	T
T	F	F	F	T	F
T	T	F	T	T	T

Example: IF (you are hungry OR you are thirsty) THEN you visit the cafeteria

$$(p \vee q) \rightarrow r \qquad \text{where} \qquad \begin{array}{l} p : \text{you are hungry} \\ q : \text{you are thirsty} \\ r : \text{you visit the cafeteria} \end{array}$$

- *You are thirsty:* q is T. In both cases r is T.
  (you visit the cafeteria)
- *You did visit the cafeteria:* r is T.
  Are you hungry? We don't know.
  Are you thirsty? We don't know.
  (You accompanied your hungry friend (row 2).)
- *You did not visit the cafeteria:* r is F.
  p and q are both F.
  (You are neither hungry nor thirsty.)

	p	q	r	$\boldsymbol{(p \vee q) \rightarrow r}$
1.	F	F	F	T
2.	F	F	T	T
3.	F	T	F	F
4.	F	T	T	T
5.	T	F	F	F
6.	T	F	T	T
7.	T	T	F	F
8.	T	T	T	T

Equivalent Compound Statements

p	q	$p \rightarrow q$	$\neg q \rightarrow \neg p$	$\neg p \vee q$	$q \rightarrow p$
F	F	T	T	T	T
F	T	T	T	T	F
T	F	F	F	F	T
T	T	T	T	T	T
		rains → wet grass	dry grass → no rain	no rain ∨ wet grass	wet grass → rain

$$p \rightarrow q \overset{\text{eqv}}{\equiv} \neg q \rightarrow \neg p \overset{\text{eqv}}{\equiv} \neg p \vee q$$

Order is very important: $p \rightarrow q$ and $q \rightarrow p$ **do not** mean the same thing.

IF I'm dead, THEN my eyes are closed **vs.** IF my eyes are closed, THEN I'm dead

Pop Quiz 3.5. Compound propositions are used for program control flow, especially IF...THEN....

```
if(x > 0 || (y > 1 && x < y))
  Execute some instructions.
```

```
if(x > 0 || y > 1)
  Execute some instructions.
```

Use truth-tables to show that both do the same thing. Which do you prefer and why?

Proving an Implication: Reasoning Without Facts

IF (n^2 is even) THEN (n is even).

p : n^2 is even
q : n is even

$p \to q$

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

What is n? How to prove?

We must show that the highlighted row *cannot* occur.

In this row, q is F: $n = 2k + 1$.

$$n^2 = (2k+1)^2 = 2(2k^2 + 2k) + 1$$

p *cannot* be T. This row cannot happen: $p \to q$ is always T. ■

Quantifiers

EVERY person has A soulmate.

Kilam has some gray hair.
Everyone has some gray hair.
Any map can be colored with 4 colors with adjacent countries having different colors.
Every even integer $n > 2$ is the sum of 2 primes *(Goldbach, 1742)*.
Someone broke this faucet.
There exists a creature with blue eyes and blonde hair.
All cars have four wheels.

These statements are more complex because of *quantifiers*:

EVERY; A; SOME; ANY; ALL; THERE EXISTS.

Compare:

My Ford Escort has four wheels;
ALL cars have four wheels.

Predicates Are Like Functions

ALL cars have four wheels

Define *predicate* $P(c)$ and its *domain*

$$\begin{aligned} C &= \{c \mid c \text{ is a car}\} && \leftarrow \text{ set of cars} \\ P(c) &= \text{"car } c \text{ has four wheels"} \end{aligned}$$

"for all c in C, the statement $P(c)$ is true."

$$\forall c \in C : P(c).$$

($\forall$ means "for all")

	Predicate	Function
	$P(c)$ = "car c has four wheels"	$f(x) = x^2$
Input	parameter $c \in C$	parameter $x \in \mathbb{R}$
Output	**statement** $P(c)$	**value** $f(x)$
Example	P(Jen's VW) = "car 'Jen's VW' has four wheels"	$f(5) = 25$
	$\forall c \in C : P(c)$	$\forall x \in \mathbb{R},\ f(x) \geq 0$
Meaning	For all $c \in C$, the statement $P(c)$ is T.	For all $x \in \mathbb{R}$, $f(x)$ is ≥ 0.

There EXISTS a Creature with Blue eyes and Blonde Hair

Define *predicate* $Q(a)$ and its *domain*

$$\begin{aligned} A &= \{a \mid a \text{ is a creature}\} && \leftarrow \text{ set of creatures} \\ Q(a) &= \text{“}a \text{ has blue eyes and blonde hair”} \end{aligned}$$

“there exists a in A for which the statement $Q(a)$ is true.”

$$\exists a \in A : Q(a).$$

($\exists$ means “there exists”)

$$\begin{aligned} G(a) &= \text{“}a \text{ has blue eyes”} \\ H(a) &= \text{“}a \text{ has blonde hair”} \end{aligned}$$

$$\exists a \in A : \underbrace{(G(a) \wedge H(a))}_{\text{compound predicate}}$$

(When the domain is understood, we don't need to keep repeating it. We write $\exists a : Q(a)$, or $\exists a : (G(a) \wedge H(a))$.)

Negating Quantifiers

IT IS NOT THE CASE THAT(There is creature with blue eyes and blonde hair)

Same as: "All creatures don't have blue eyes and blonde hair"

$$\neg\Big(\exists a \in A : Q(a)\Big) \quad \stackrel{\text{eqv}}{\equiv} \quad \forall a \in A : \neg Q(a)$$

IT IS NOT THE CASE THAT(All cars have four wheels)

Same as: "There is a car which does not have four wheels"

$$\neg\Big(\forall c \in C : P(c)\Big) \quad \stackrel{\text{eqv}}{\equiv} \quad \exists c \in C : \neg P(c)$$

When you take the negation inside the quantifier and negate the predicate, you must switch quantifiers: $\forall \to \exists$, $\exists \to \forall$

Every Person Has a Soul Mate

Define domains and a predicate.

$$A = \{a \mid a \text{ is an person}\}.$$

$$P(a,b) = \text{“Person } a \text{ has as a soul mate person } b\text{.”}$$

- There is some special person b who is a soul mate to every person b.

$$\exists b : \big(\forall a : P(a,b)\big).$$

- For every person a, they have there own personal soul mate b.

$$\forall a : \big(\exists b : P(a,b)\big).$$

When quantifiers are mixed, the order in which they appear is important for the meaning. Order generally cannot be switched.

Proofs with Quantifiers

Claim 1. $\forall n > 2$: IF n is even, THEN n is a sum of two primes. *(Goldbach, 1742)*

Claim 2. $\exists (a, b, c) \in \mathbb{N}^3 : a^2 + b^2 = c^2$. ($(a, b, c) \in \mathbb{N}^3$ means triples of natural numbers)

Claim 3. $\neg\exists (a, b, c) \in \mathbb{N}^3 : a^3 + b^3 = c^3$.

Claim 4. $\forall (a, b, c) \in \mathbb{N}^3 : a^3 + b^3 \neq c^3$.

Think about what it would take to prove these claims.