FINAL: 180 Minutes

Last Name: Solution
First Name: 
RIN: 
Section: 

Answer ALL questions.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

You MUST show CORRECT work to get full credit.

When in doubt, TINKER.

<table>
<thead>
<tr>
<th>1-12</th>
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<td>10 each</td>
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1 Determine the type of proof and prove: a right triangle with integer sides cannot be isosceles.

**Contradiction.**

Let there be an isosceles right triangle with integer sides of lengths \(a\) and \(c\).

\[
\begin{array}{c}
\text{a} \\
\text{a} \\
\text{c}
\end{array}
\]

Then \(a^2 + a^2 = c^2\), so \(2a^2 = \left(\frac{c}{a}\right)^2\), or \(\sqrt{2a} = \frac{c}{a}\).

This is a contradiction, as \(\sqrt{2a}\) is irrational. Therefore no such triangle exists.
Prove that the product of any 5 consecutive natural numbers is divisible by 5!.

Let \( n, n+1, \ldots, n+4 \) be 5 consecutive natural numbers.

By the pigeon hole principle, every 5 consecutive numbers contain a multiple of 5, because the multiples of 5 are within distance 5 of each other. This means one of \( n, n+1, \ldots, n+4 \) is divisible by 5.

Similarly, multiples of 3 are within distance 3 of each other, so one of \( n, n+1, \ldots, n+4 \) is divisible by 3.

Similarly, multiples of 4 are within distance 4 of each other, so one of \( n, n+1, \ldots, n+4 \) is divisible by 4. There is at least one other even number in the list \( n, n+1, \ldots, n+4 \), as there are at least two even numbers in every list of five numbers.

Altogether, we can conclude that the product of \( n, n+1, \ldots, n+4 \) is divisible by \( 5 \cdot 4 \cdot 3 \cdot 2 = 5! \).
Alternatively:

Write the consecutive numbers as $n+1,...,n+5$. Then

$$(n+1)\ldots(n+5) = \frac{(n+5)!}{n!} = 5! \left[ \frac{(n+5)!}{n!5!} \right]$$

$$= 5! \left( \frac{n+5}{n} \right)$$

so we see directly that the product is divisible by $5!$. 


Let $G_0 = 0$, $G_1 = 1$, and $G_n = 7G_{n-1} - 12G_{n-2}$ for $n > 1$. Compute $G_5$. Show $G_n = 4^n - 3^n$ for $n \geq 0$.

Let $G_0 = 0$, $G_1 = 1$, and $G_n = 7G_{n-1} - 12G_{n-2}$ for $n > 1$. Compute $G_5$. Show $G_n = 4^n - 3^n$ for $n \geq 0$.

\[ G_2 = 7G_1 - 12G_0 = 7 \]
\[ G_3 = 7G_2 - 12G_1 = 7 \cdot 7 - 12 \cdot 1 = 37 \]
\[ G_4 = 7G_3 - 12G_2 = 7 \cdot 37 - 12 \cdot 7 = 7 \cdot 25 = 175 \]
\[ G_5 = 7G_4 - 12G_3 = 7 \cdot 175 - 12 \cdot 37 = 1225 - 444 = 781 \]
\[ G_5 = 781 \]

Proof. $G_n = 4^n - 3^n$ for $n \geq 0$.

We use strong induction.

When $n = 0$, $G_0 = 0 = 4^0 - 3^0$, so the formula holds.

Now assume that $G_k = 4^k - 3^k$ when $k \geq 0$ and $k \leq n$.

Then
\[ G_{n+1} = 7G_n - 12G_{n-1} = 7 \cdot (4^n - 3^n) - 12 \cdot (4^{n-1} - 3^{n-1}) \]

by the inductive hypothesis, so
\[ G_{n+1} = 7 \cdot (4^n - 3^n) - (3 \cdot 4^n - 4 \cdot 3^n) \]
\[ = 4 \cdot 4^n - 3 \cdot 3^n \]
\[ = 4^{n+1} - 3^{n+1} \]

so the formula holds for $n+1$.

We conclude that $G_n = 4^n - 3^n$ for $n \geq 0$ by the principle of induction.
4. Determine and prove the order-relationships between $\ln n$, $\ln(n^2 + 1)$, and $\ln(2n)$.

- $\ln(2n) = \ln(2) + \ln(n) \in \Theta(\ln n)$

- $\ln(n^2 + 1) \in \Theta(\ln(n^2))$ and $\ln(n^2) = 2\ln(n)$, so $\ln(n^2 + 1) \in \Theta(\ln n)$

Thus all three functions are asymptotically equivalent ($\Theta$)
Prove that
\[ x^k \equiv y^k \mod d \]
implies that
\[ x \equiv y \mod \left( \frac{d}{\gcd(k, d)} \right). \]

\textbf{Proof}

Assume \( x^k - y^k \equiv 0 \mod d \); this means \( d \mid (x-y)^k \). Now observe that both \( d \) and \( (x-y)^k \) are divisible by \( \gcd(d, k) \), so
\[
\frac{d}{\gcd(d, k)} \mid \frac{(x-y)^k}{\gcd(d, k)}. \]

Also note that by the definition of the \( \gcd \), the numbers \( \frac{d}{\gcd(d, k)} \) and \( \frac{k}{\gcd(d, k)} \) are coprime.

Since \( \frac{d}{\gcd(d, k)} \mid \frac{(x-y)^k}{\gcd(d, k)} \) and does not divide \( \frac{k}{\gcd(d, k)} \), we conclude that
\[
\frac{d}{\gcd(d, k)} \mid (x-y),
\]
so
\[ x \equiv y \mod \left( \frac{d}{\gcd(d, k)} \right). \]
Adam, Barb, Charlie, and Doris each independently choose a random number uniformly distributed in \{1, 2, 3, 4, 5\}. What is the probability that some pair chooses the same number? What if there are \(k\) people and \(n\) numbers?

\[
P[\text{some pair chooses the same number}] = 1 - P[\text{everyone chooses a unique number}]
\]

\[
= 1 - \frac{(5\, 4)!}{5^4}
\]

\[
\frac{\text{number of ways to assign four unique numbers of 5 possible numbers, to four people}}{\text{number of ways to assign each person a number in 1 through 5.}}
\]

In the general case,

\[
P[\text{some pair chooses the same number}] = 1 - \frac{(k\, k)!}{n^k}
\]
Let $\mathcal{V}$ be a set of $n$ vertices, and let the edge set $\mathcal{E}$ be initially empty. For each pair of vertices $i \neq j$, add the edge $(i, j)$ to $\mathcal{E}$ with probability $p$. Give the pdf for the degrees of the nodes of this graph.

Each node is connected to each other node w/ prob $p$. Let $X$ be the degree of a node, then

$$X \sim \text{Binomial}(n-1, p)$$

so the pdf is given by

$$P[X = k] = \binom{n-1}{k} p^k (1-p)^{n-1-k}$$

for $k \in \{0, \ldots, n-1\}$.
Voltage in the US has a mean of 120V and a standard deviation of 5V. A device's operating voltage is 112–128. Use Chebyshev's inequality to bound the probability that the device will not be damaged when turned on.

The probability the device will not be damaged is

\[ P \left( V \in [112, 128] \right) = 1 - P \left( V \notin [112, 128] \right) \]

\[ = 1 - P \left| V - 120 \right| > 8 \]

where \( V \) is the voltage. Now applying Chebyshev:

\[ P \left( \left| V - 120 \right| > 8 \right) = P \left( \left| V - EV \right|^2 > 64 \right) \]

\[ = P \left( \left| V - EV \right|^2 > \frac{64}{25} \cdot 25 \right) \]

\[ = P \left( \left| V - EV \right|^2 > \frac{64}{25} \cdot \sigma^2(V) \right) \]

\[ \leq \frac{1}{\left( \frac{64}{25} \right)} \]

so

\[ P \left[ \text{device not damaged} \right] \geq 1 - \frac{25}{64} \]
Give a DFA for strings whose even digits alternate between 0 and 1.
For CFG $S \rightarrow 0S|S1|01$, prove that no string has 10 as a substring.

\textbf{Proof:}

We use induction on the length of the strings generated by the CFG (denoted $\text{len}(s)$).

When $\text{len}(s) = 1$, the strings cannot have 10 as a substring.

Assume no strings of length $\ell$ derived from the CFG have 10 as a substring. Let $s$ be a string of length $\ell + 1$ derived from the CFG, then $s = 0w$ or $s = w1$ for a string $w$ of length $\ell$ derived from the CFG. By the inductive assumption $w$ does not contain 10 as a substring. Since adding a 0 at the start or a 1 at the end of $w$ does not introduce a 10 substring, $s$ does not contain 10 as a substring.

We conclude by the principle of induction that no string derived from this CFG contains 10 as a substring.
Consider the language of palindromes $\mathcal{L} = \{ \omega | \omega \in \{0,1\}^*, \omega = \omega^R \}$. Give well-written high-level pseudocode for a decider for this language.

1. Return to $\ast$
2. Go right to the first unmarked input. Mark and remember it. If there are no unmarked digits before you reach $\omega$, halt and ACCEPT.
3. Go right to $\omega$, then go left to the first unmarked input. If it matches the remembered digit, mark it and GOTO 1. Otherwise, halt and REJECT.
Given an ultimate-debugger $D$ that takes $(M)\#\omega$ and decides if TM $M$ halts on input $\omega$, show that every recognizer of a language $L$ can be converted into a decider for the language.

Let $R$ be a recognizer for $L$. Take $\Delta$ to be the Turing Machine that computes

$$\Delta(\omega) = \begin{cases} R(\omega) & \text{if } D(<R>\#\omega) = \text{ACCEPT} \\ \text{REJECT} & \text{if } D(<R>\#\omega) = \text{REJECT} \end{cases}$$

By construction,

1) If $\omega \in L$, then $R(\omega)$ halts and ACCEPTS,
so $D(<R>\#\omega) = \text{ACCEPT}$, so
$$\Delta(\omega) = R(\omega) = \text{ACCEPT}$$

2) if $\omega \notin L$, then either $R(\omega)$ does not halt,
in which case $D(<R>\#\omega) = \text{REJECT}$, so
$$\Delta(\omega) = \text{REJECT},$$
or $R(\omega)$ halts and REJECTS, so
$$D(<R>\#\omega) = \text{ACCEPT},$$
so
$$\Delta(\omega) = R(\omega) = \text{REJECT}.$$  

That is,

$$\Delta(\omega) = \begin{cases} \text{ACCEPT} & \text{if } \omega \in L \\ \text{REJECT} & \text{if } \omega \notin L \end{cases}$$

so $\Delta$ is a decider for $L$.  

\[\square\]
SCRATCH