Final Exam

170 Minutes

First Name: ____________________

Last Name: ____________________

RIN: _________________________

NO COLLABORATION or electronic devices.
Any violations will result in an F.
No questions allowed during the test unless you think there is a mistake.

GOOD LUCK!

10 points per correct multiple-choice answer. Circle exactly one answer.
20 points per correct answer to Problems 2-6.
You MUST show CORRECT work to get credit.
Correct answers with no explanation will get a 0.

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1. What is the asymptotic behavior of the sum $S(n) = \sum_{i=1}^{n} i^9 + 100i^8$?

A. $O(n^9)$
B. $O(n^8)$
C. $O(100n^9)$
D. $O(n^{10})$
E. None of the above.

2. What can we say about this statement: $\exists C > 0 : \forall n \geq 1 : n^2 < \frac{1}{2}Cn^2 + n$?

A. True
B. False
C. Depends on $C$
D. Depends on $n$
E. None of the above.

3. What can we say about this statement: $\exists C > 0 : \forall n \geq 1 : n^3 < \frac{1}{2}Cn^2 + n$?

A. True
B. False
C. Depends on $C$
D. Depends on $n$
E. None of the above.

4. Consider the recurrence $T_1 = 1, T_n = T_{n-1} + n^3$. Estimate $T_{10}$:

A. 250
B. 2500
C. 25000
D. 250000
E. None of the above.

5. Calculate the sum $\sum_{i=1}^{5} \sum_{j=0}^{3} i2^j$.

A. 200
B. 225
C. 250
D. 275
E. None of the above.
6. What is the last digit of $3^{100}$?

A 1  
B 3  
C 5  
D 7  
E 20

7. Consider a graph $G$ where every vertex has degree 2. What do we know?

A $G$ must have at least 3 vertices.  
B $G$ can be a cycle (i.e., the entire graph can be a single cycle).  
C $G$ must have a cycle.  
D $G$ is not a tree.  
E All of the above.

8. Consider a graph $G$ where every vertex has degree 3. What do we know?

A $G$ must have an even number of vertices.  
B $G$ can be a tree.  
C $G$ cannot have any cycles.  
D $G$ cannot exist.  
E All of the above.

9. How many graphs with 5 vertices are there?

A 32  
B 128  
C 256  
D 1024  
E None of the above.

10. Suppose FOCS has 300 students, and I split them in 3 sections with 100 students each. How many ways to split the students are there?

A $\binom{300}{100}$  
B $\binom{300}{200}$  
C $\binom{300}{100,100,100}$  
D $300!$  
E None of the above.
11. You flipped 4 fair coins. What is the probability of exactly 3 heads?

A 1/16
B 2/16
C 3/16
D 4/16
E None of the above.

12. You flipped 4 fair coins. What is the probability of exactly 3 heads, given that the first coin was H?

A 1/8
B 2/8
C 3/8
D 4/8
E None of the above.


A 10
B 16
C 22
D 28
E None of the above.

14. Let $X$ be a positive random variable. If $\sigma^2(X) = 3$ and $E[X^2] = 5$, what is $E[X]$?

A $\sqrt{2}$
B 2
C 3
D 4
E None of the above.

15. On any day, with probability 0.5 I go to the cafeteria at noon: if I go at noon, I get my food at 12:10pm with probability 0.2 and at 12:30pm with probability 0.8; with probability 0.5, I go to the cafeteria at 12:30pm and get my food at 12:30pm. How many days am I expected to wait until I get my food at 12:10pm (start counting from 1)?

A 5
B 10
C 15
D 20
E None of the above.
16. What do we know about the language $L = \{w\#w \mid w \in \{0,1\}^*\}$?
   
   A. It is regular.
   B. It is context-free.
   C. It is decidable.
   D. It is undecidable.
   E. None of the above.

17. What do we know about the language $L = \{w\#w^R \mid w \in \{0,1\}^*\}$?
   
   A. It is regular.
   B. It is context-free.
   C. It is decidable.
   D. It is undecidable.
   E. None of the above.

18. Can a pushdown automaton (PDA) solve the language $L = \{ww^R \mid w \in \{0,1\}^*\}$?
   
   A. No, the language is not context-free.
   B. Yes, a deterministic PDA can solve $L$.
   C. Yes, a non-deterministic PDA can solve $L$.
   D. No, the language is not decidable.
   E. None of the above.

19. Which of the following problems is decidable?
   
   A. The halting problem.
   B. Deciding whether a given program will print “Hello World”.
   C. Deciding whether a given program will terminate.
   D. Deciding whether a given C program is well-formatted.
   E. None of the above.

20. What do we know about the traveling salesman problem?
   
   A. It is known to be in the class P.
   B. I can solve it by listing all possible trajectories.
   C. It is undecidable.
   D. I can always list all trajectories in linear time.
   E. None of the above.
Problem 2. Prove that \( \forall n \geq 1, \log_2(n) \leq n \).
Problem 3. Suppose you are determined to flip a fair coin until you get the sequence HHH. What is the expected number of flips?
Problem 4. Prove that for any $n \geq 3$, there exists a set of $n$ distinct natural numbers $x_1, \ldots, x_n$ such that each $x_i$ divides the sum $s = x_1 + \cdots + x_n$, i.e., $s = x_i k_i$ for some $k_i \in \mathbb{N}$. Tinker, tinker, tinker.
Problem 5. Consider the language of all odd-length zero strings \( L_O = \{0, 00, 0000, \ldots \} \). Prove that \( L_O \) has an undecidable subset.
Problem 6. Consider the language $\mathcal{L}_{AddTwo} = \{0^n 1^{n+2} \}$. Give pseudocode for a Turing Machine that decides this language.

[Your pseudocode needs to be detailed enough so it is clear that each step can indeed be performed using a Turing Machine.]
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