MIDTERM: 90 Minutes

Last Name: Solutions
First Name: 
RIN: 
Section: 

Answer ALL questions. You may use one double sided $8\frac{1}{2} \times 11$ crib sheet.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.
You MUST show CORRECT work, even on multiple choice questions, to get credit.

GOOD LUCK!

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INSTRUCTIONS

1. This is a closed book test. No electronics, books, notes, internet, etc.
2. One 8.5 x 11 double sided crib sheet is allowed.
3. The test will become available in Submitty at 8am on the test date.
4. Your PDF is due in Submitty by 2pm.
5. By submitting the test you attest that:
   – the work is entirely your own.
   – you obeyed the time limits of the exam.
6. Your submission must be typed and submitted as a PDF file.
7. The first page should list your 15 multiple choice answers, like:

   (1) A
   (2) B
   (3) C
   (4) D
   ...
   (15) A

8. Start each of problems 2–6 on a new page. SHOW WORK.
9. After the answers, start a new page to show work for the multiple choice:

   (1) Because z is even
   (2) Because $\sqrt{2}$ is irrational.
   (3) Number of links is
       \[1 + 2 + \cdots + 10 = 55\]
       ...
   (15) Because we proved in class that $\ell = n - 1$

   – Some problems may be “easy”, so give a one line justification.
   – Some problems may require a detailed reasoning.
   – correct answers: 10 points
   – wrong answers or no work/explanation: 0.

10. If you don’t show correct work, you won’t get credit.
11. Submit with plenty of time to spare. A late test won’t be accepted.
    – We won’t accept submissions that are even 1 second late.
1. Circle one answer per question. 10 points for each correct answer.

(1) What is the correct asymptotic behavior (order analysis) for the sum $S(n) = \sum_{i=1}^{n} \sqrt{i}$.

A $S(n) \in \Theta(n)$. [X]
B $S(n) \in \Theta(n^2)$. [X]
C $S(n) \in \Theta(n^3)$. [X]
D $S(n) \in \Theta(n^4)$. [X]
E None of the above.

(2) What is the correct asymptotic behavior (order analysis) for the sum $S(n) = \sum_{i=1}^{n^2} i$.

A $S(n) \in \Theta(n)$. [X]
B $S(n) \in \Theta(n^2)$. [X]
C $S(n) \in \Theta(n^3)$. [X]
D $S(n) \in \Theta(n^4)$. [X]
E None of the above.

(3) Which is the correct asymptotic order relationship that describes the sum $S(n) = \sum_{i=0}^{n} 2^i$.

A $S(n) \in \Theta(n^2)$. [X]
B $S(n) \in \Theta(2^n)$. [X]
C $S(n) \in \omega(2^n)$. [X]
D $S(n) \in o(2^n)$. [X]
E None of the above.

(4) Estimate $\ln(10^{9!})$, that is the logarithm of the factorial of $10^9$.

A $2 \times 10^{10}$. [X]
B $2 \times 10^{20}$. [X]
C $2 \times 10^{50}$. [X]
D $2 \times 10^{200}$. [X]
E $2 \times 10^{400}$. [X]

(5) $\gcd(210, 385) = 210x + 385y$ where $x, y \in \mathbb{Z}$. What are a possible choice for $x, y$?

A $x = 1, \quad y = 1$. [X]
B $x = -1, \quad y = 1$. [X]
C $x = 2, \quad y = -1$. [X]
D $x = 1, \quad y = 1$. [X]
E None of the above.

\[ \gcd(210, 385) = \gcd(115, 210) = \gcd(35, 175) = \gcd(0, 35) \]

\[ x = 2, \quad y = -1 \]
(6) What is the remainder when $29^{2019} - 22^{2019}$ is divided by 3?

\[ \begin{align*} 
29 & \equiv 2 \\
(29^2) & \equiv 1 \\
(29^3) & \equiv 1 \\
29^{2019} & \equiv 29 \equiv 2. \\
22 & \equiv 1 \\
22^{2019} & \equiv 22 \equiv 2 \equiv 2-1 \equiv 1. 
\end{align*} \]

\[ \text{E None of the above.} \]

(7) A friendship network (simple graph) has vertices having degree sequence $\delta = [5, 4, 3, 2, 2]$. How many edges (friendship links) are in this friendship network?

\[ \begin{align*} 
\text{A} & : 6 \text{ edges} \\
\text{B} & : 7 \text{ edges} \\
\text{C} & : 8 \text{ edges} \\
\text{D} & : \text{Not enough information to determine the number of edges} \\
\text{E} & : \text{This friendship network cannot possibly exist} 
\end{align*} \]

(8) You wish to color the graph on the right so that linked vertices do not get the same color. What is the minimum number of colors needed (the chromatic number).

\[ \begin{align*} 
\text{A} & : 2 \\
\text{B} & : 3 \\
\text{C} & : 4 \\
\text{D} & : 5 \\
\text{E} & : 6 
\end{align*} \]

(9) At a party with $n$ people, everyone shakes hands with everyone else. How many handshakes occur?

\[ \begin{align*} 
\text{A} & : \frac{1}{2}n(n-1) \\
\text{B} & : \frac{1}{2}n(n+1) \\
\text{C} & : 2n \\
\text{D} & : n^2 \\
\text{E} & : \text{None of the above.} 
\end{align*} \]

(10) A connected graph has $n \geq 2$ vertices and no cycles. Does the graph have a degree 1 vertex?

\[ \begin{align*} 
\text{A} & : \text{Yes, always.} \\
\text{B} & : \text{No, never.} \\
\text{C} & : \text{If } n \text{ is even, yes, otherwise no.} \\
\text{D} & : \text{If } n \text{ is even no, otherwise yes.} \\
\text{E} & : \text{None of the above.} 
\end{align*} \]
(11) How many subsets of \{1, 2, 3, 4, 5, 6, 7\} contain one even number?

- A \[3 + 2^4.\]
- B \[3 \times 2^4.\]
- C \[\binom{7}{3}.\]
- D \[2^7.\]
- E None of the above.

- Choose even \(3\) ways
- Choose subset of \(2^4\) ways
Odd numbers
Product Rule \[3 \times 2^4\]

(12) There are 4 candy colors. How many goody-bags with 10 candies can you make.

- A \[10^4.\]
- B \[4^{10}.\]
- C \[4 \binom{10}{4}.\]
- D \[\binom{14}{4}.\]
- E None of the above.

\[
\begin{align*}
\binom{n+k-1}{k-1} &= \binom{13}{3} \\
\text{Problem}\end{align*}
\]

(13) How many sequences of four non-negative integer solutions add up to 10? (e.g. \(4, 3, 2, 1\), \((3, 4, 2, 1)\)).

- A \[10^4.\]
- B \[4^{10}.\]
- C \[\binom{10}{4}.\]
- D \[\binom{14}{4}.\]
- E None of the above.

\[
\text{Problem}\text{Same as }\binom{13}{3}
\]

(14) In how many ways can you misspell TEDDY, assuming you use all the same letters?

- A \[57.\]
- B \[58.\]
- C \[59.\]
- D \[60.\]
- E None of the above.

\[
\text{Anagrams } = \frac{5!}{1!} = 60 \\
\text{miss spell } \rightarrow 2^4 \times 9 \text{ ways } \left[\frac{60-17}{9}\right]
\]

(15) How many of the 1,000 numbers in \{0, 1, \ldots, 999\} contain the digits 1 or 2?

- A \[460.\]
- B \[474.\]
- C \[488.\]
- D \[512.\]
- E None of the above.

\[
\frac{\text{All } 3\text{-digit strings}}{\text{Do not contain 1 or 2}} = \frac{9 \times 9 \times 9}{7 \times 8} = 7 \times 92
\]

\[
\text{Such numbers one 8 digits } \rightarrow 8^3 - 8^2 = 488 \\
\text{Contain 1 or 2 } = 10^3 - 8^3 = 488 \\
\text{488} \div 2^9
\]
Prove that \( \log_2 9 \) is not a rational number.

Proof by contradiction:

Assume \( \log_2 9 = \frac{a}{b} \)

\[ \Rightarrow \quad 9 = 2^{a/b} \]

\[ \Rightarrow \quad 9^b = (2^{a/b})^b = 2^a \]

\[ \uparrow \quad \text{odd} \quad \uparrow \quad \text{even} \]

Contradiction

\[ \therefore \quad \log_2 9 \neq \frac{a}{b}. \]
Prove by induction for all \( n \geq 1 \): \[ \sum_{i=1}^{n} i2^i = (n-1)2^{n+1} + 2. \]

**Proof by Induction**

**Base Case** \( P(1) \): \[ \sum_{i=1}^{1} i2^i = 2^1 = 2 \] Base case is true.

**Induction Step**

Assume \( \sum_{i=1}^{n} i2^i = (n-1)2^{n+1} + 2 \).

Prove \( \sum_{i=1}^{n+1} i2^i = n2^{n+2} + 2 \).

\[
\begin{align*}
\sum_{i=1}^{n+1} i2^i &= \sum_{i=1}^{n} i2^i + (n+1)2^{n+1} \\
&= (n-1)2^{n+1} + 2 + (n+1)2^{n+1} \\
&= (n-1+n+1)2^{n+1} + 2 \\
&= 2n2^{n+1} + 2 \\
&= n2^{n+2} + 2
\end{align*}
\]

Proving \( P(n+1) \) by induction.

\( P(n) \) is true for all \( n \geq 1 \). \( \Box \)
Find a formula for $A_n$ and prove your answer. $A_0 = 1$ and $A_n = A_{n-1} + n$ for $n \geq 1$

\[
\begin{align*}
A_n &= A_{n-1} + n \\
A_{n-1} &= A_{n-2} + n-1 \\
A_{n-2} &= A_{n-3} + n-2 \\
\vdots \\
A_2 &= A_1 + 2 \\
A_1 &= A_0 + 1 \\
A_0 &= 1 \\
\end{align*}
\]

Add:

\[
A_n = n + n-1 + \cdots + 1 + 1 = 1 + \underbrace{(1+2+3+\cdots+n)}_{\text{common sum}} = 1 + \frac{1}{2}n(n+1)
\]

Proof by Induction

Base: $A_0 = 1 + \frac{1}{2} \cdot 0 \cdot (0+1) = 1$ (✓)

Induction Step: Assume $A_n = 1 + \frac{1}{2}n(n+1)$

Prove $A_{n+1} = 1 + \frac{1}{2} (n+1)(n+2)$

\[
\begin{align*}
A_{n+1} &= A_n + n+1 \\
&= 1 + \frac{1}{2} n(n+1) + n + 1 \\
&= 1 + \left(\frac{1}{2} n + 1\right)(n+1) \\
&= 1 + \frac{1}{2} (n+1)(n+2) \quad (✓)
\end{align*}
\]

By induction, $A_n = 1 + \frac{1}{2} n(n+1)$ for all $n \geq 0$. (✓)
How many subsets of \( \{1, 2, 3, \ldots, 10\} \) have an even sum. (The empty set \( \emptyset \) has even sum.)

For example, the subset \( \{2, 6\} \) has even sum but \( \{2, 7\} \) has odd sum.

(1) 5 even \#’s and 5 odd \#’s.

\[
\begin{array}{c}
\text{pick any subset of} \\
\text{25 ways}
\end{array}
\]

\[
\begin{array}{c}
\text{pick an even number} \\
\text{16}
\end{array}
\]

\[
\begin{array}{c}
\text{\( \binom{5}{3}, \binom{5}{2}, \binom{5}{1} \)} \\
\text{\( 1 \times 10 + 5 \)}
\end{array}
\]

\[
\begin{array}{c}
\text{\( 2 \times 16 = 2^9 = 512 \)}
\end{array}
\]

(2) \[
\text{Sum } = 1 + 2 + 3 + \ldots + 10 = \frac{10 \times 11}{2} = 55 \text{ is odd.}
\]

If you pick an even subset, the complement subset is odd.

\[
\begin{array}{c}
\text{every even subset} \iff \text{corresponding odd subset}
\end{array}
\]

\[
\begin{array}{c}
\# \text{even subsets} = \# \text{odd subsets}
\end{array}
\]

\[
\begin{array}{c}
\# \text{even} + \# \text{odd} = \text{All subsets} = 2^{10},
\text{\( 2 \times \# \text{even} \)}
\end{array}
\]

\[
\begin{array}{c}
\# \text{even} = \frac{2^{10}}{2} = 2^9 = 512
\end{array}
\]

(3) \[
\begin{array}{c}
E(n) = \text{even subsets with } \{1, \ldots, n\}
\end{array}
\]

\[
\begin{array}{c}
O(n) = \text{odd subsets with } \{1, \ldots, n\}
\end{array}
\]

You can either choose \( n \) or not.

\[
\begin{array}{c}
\forall n \geq 0, E(n) = \left\{ \begin{array}{ll}
2E(n-1) & \text{\( n \) even} \\
O(n-1) + E(n-1) & \text{\( n \) odd}
\end{array} \right.
\end{array}
\]

\[
\begin{array}{c}
E(n) \text{ for } n \geq 1
\end{array}
\]

\[
\begin{array}{c}
0(0) = \left\{ \begin{array}{ll}
2O(n-1) & \text{\( n \) even} \\
O(n-1) + E(n-1) & \text{\( n \) odd}
\end{array} \right.
\end{array}
\]

\[
\begin{array}{c}
o(1) = 1.
\end{array}
\]

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FOCSbook has \( n \) people and each person has \( n - 2 \) friends. Find all \( n \) where it is possible? Precisely state your result and prove it. TINKER!

**Claim**

- This is possible when \( n \) is even.
- Not possible when \( n \) is odd.

**Proof**

Look at \( f \) enemies. Each person has 1 enemy.

- Enemy Network has \( n \)-people each
- With 1 enemy

- If \( n \) is odd, degree sum is \( n - 1 \) (odd)

- Not possible by handshaking theorem.

If \( n \) is even, make \( \frac{n}{2} \) pairs

and each pair are enemies.

\[
\begin{array}{cccccc}
& & & & \\
& & & & \\
\end{array}
\]

- Enemy network for even \( n \).

\[\square\]