MIDTERM: 90 Minutes

Last Name: Solutions
First Name: 
RIN: 
Section: 

Answer ALL questions. You may use one double sided $8\frac{1}{2} \times 11$ crib sheet.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.
You MUST show CORRECT work, even on multiple choice questions, to get credit.

GOOD LUCK!

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1. Circle one answer per question. 10 points for each correct answer.

(1) Compute \( S = \sum_{i=1}^{2} \sum_{j=1}^{4} (i + j). \)

\[ S = 2 \sum_{i=1}^{2} i + 2 \sum_{j=1}^{4} j = 4 \left( \frac{2 \cdot 1}{2} \right) + 2 \left( \frac{4 \cdot 5}{2} \right) = 12 + 20 = \boxed{32} \]

A) \( S = 28. \)
B) \( S = 30. \)
C) \( S = 32. \)
D) \( S = 34. \)
E) None of the above.

(2) Estimate. Which approximation is closest to \( S = \sum_{i=1}^{n} i \).

\[ \frac{n(n+1)}{2} \approx \frac{2^{100}}{2} = \left( \frac{2^{10}}{2} \right)^{10} = \left( 10^{2} \right)^{10} = \sqrt[10]{1 \times 10^{30}} \]

A) \( S \approx 1275. \)
B) \( S \approx 10^{10}. \)
C) \( S \approx 10^{20}. \)
D) \( S \approx 10^{30}. \)
E) \( S \approx 10^{40}. \)

(3) What is the correct asymptotic behavior (order analysis) for the function \( S(n) = n\sqrt{n}. \)

\[ n\sqrt{n} = n^{\frac{3}{2}} \]

A) \( S(n) \in \Theta(n). \) \( \times \)
B) \( S(n) \in \Theta(n^2). \) \( \times \)
C) \( S(n) \in \Theta(n^3). \) \( \times \)
D) \( S(n) \in \Theta(n^4). \) \( \times \)
E) None of the above.

(4) What is the correct asymptotic behavior (order analysis) for the sum \( S(n) = \sum_{i=0}^{n} 2^i. \)

\[ \sum_{i=0}^{n} 2^i = 2^{n+1} - 1 \]

A) \( S(n) \in \Theta(2^n). \)
B) \( S(n) \in \Theta((2^n)^2). \)
C) \( S(n) \in \Theta(2^{n^2}). \)
D) \( S(n) \in \Theta(2^{2n}). \)
E) None of the above.

(5) Compute \( \text{gcd}(1045, 2310). \) That is, compute the greatest common divisor of 1045 and 2310.

\[ \text{gcd}(1045, 2310) = \text{gcd}(1045, 2310 - 2 \times 1045) = \text{gcd}(1045, 2310 - 2310 - 2 \times 1045) = \text{gcd}(1045, 1045 - 4 \times 1045) = \text{gcd}(1045, 1045 - 4 \times 1045) = \text{gcd}(1045, 1045 - 4 \times 1045) = \text{gcd}(1045, 0) = 1045 \]

A) 5.
B) 10.
C) 55.
D) 95.
E) None of the above.

\[ \text{gcd}(1045, 2310) = \text{gcd}(2310, 1045) = \text{gcd}(1045, 2310 - 2 \times 1045) = \text{gcd}(1045, 0) = 55 \]

\[ \text{gcd}(1045, 2310) = \text{gcd}(55, 1045) = \text{gcd}(55, 1045 - 19 \times 55) = \text{gcd}(55, 1045 - 19 \times 55) = \text{gcd}(55, 1045 - 19 \times 55) = \text{gcd}(55, 0) = 55 \]
(6) Which degree sequence could be the degree sequence of a friendship network (simple graph)?

- **A** [5, 3, 3, 2, 1]. 5 vertices → max degree = 4
- **B** [3, 3, 2, 1]. Sum of degrees = 9 ≠ 2E contradiction
- **C** [3, 3, 3, 3].
- **D** [4, 4, 3, 2, 1]. Degree 4 vertices connected to all others → min degree ≥ 2
- **E** None of the above.

(7) You wish to color the graph on the right so that linked vertices do not get the same color. What is the minimum number of colors needed (the chromatic number).

- **A** 2
- **B** 3
- **C** 4
- **D** 5
- **E** 6

(8) Boys X, Y, Z and girls A, B, C have the preference lists shown. Which of these is a stable matching?

- **A** A–X, B–Y, C–Z.
- **B** A–X, B–Z, C–Y.
- **C** A–Y, B–X, C–Z.
- **D** A–Z, B–Y, C–X.
- **E** None of the above or there is no stable matching.

(9) What is the minimum number of children that guarantees at least two have the same first and last initial?

- **A** 26
- **B** 26 + 1
- **C** $26^2$
- **D** $26^2 + 1$
- **E** None of the above.

(10) A race has 6 runners. In how many ways can the gold, silver and bronze medal be given?

- **A** $6^3$
- **B** $inom{6}{3}$
- **C** $6 	imes 5 	imes 4$
- **D** 6!
- **E** None of the above.

Product Rule: $6 	imes 5 	imes 4$
(11) A shirt matches 2 pants. My blue tie matches 3 shirts. My red tie matches 4 shirts. How many matching outfits can I wear? (In a matching outfit, the shirt must match the tie and pants.)


(12) A shelf has some books. Alice picks 3 to read and Bob picks 4 to read. Which must be true?

A: Alice has more ways to pick her books than Bob has for picking his books.  x
B: Bob has more ways to pick his books than Alice has for picking her books.  x
C: Alice and Bob have the same number of ways for picking their books.  x
D: Alice and Bob cannot have the same number of ways for picking their books.  x
E: None of the above.

(13) How many subsets of \{1, 2, 3, 4, 5\} contain 1 and 2 or contain 3 and 4?


(14) How many different shortest paths from A to B are there?

A: 28  B: 32  C: 34  D: 36  E: None of the above.

(15) You write all 1,000 numbers in \{0, 1, \ldots, 999\}. How many times did you write the digit 1?

Prove or disprove: the integer $x$ is odd if and only if $x^2 - 1$ is divisible by 8.

If and only if means prove two implications.

Prove $8 | x^2 - 1 \implies x$ is odd.

Suppose $8 | x^2 - 1$.

$\implies x^2 - 1$ is even.

$\implies x^2$ is odd (proved in class).

$\implies x$ is odd (proved in class).  \( \Box \)

80\% for generally correct proof of one direction.

Prove $x$ is odd $\implies 8 | x^2 - 1$.

$x$ is odd $\implies x^2 - 1 = (x-1)(x+1)$

$= \text{product of consecutive even numbers.}$

$= (2k)(2k+2) = 4k(k+1)$

If $k$ is even then $4k$ is div by 8.

If $k$ is odd then $4(k+1)$ is div by 8.

In both cases $x^2 - 1$ is div by 8. \( \Box \)

100\%.

Grading

50\%: 2 implications recognized.

80\%: 1 implication generally correct.

100\%: Both implications generally correct.
For $n \in \mathbb{N}$, prove that $\text{remainder}(n, 9) = \text{remainder}(\text{sum of } n\text{'s digits}, 9)$.

E.g. 725 has remainder 5 modulo 9. The sum of the digits is 14, which also has remainder 5 modulo 9.

Basic knowledge: $a \equiv b \pmod{c} \equiv d \pmod{e}$

$\Rightarrow a+c \equiv b+d \pmod{f}$  \hspace{1cm} (1)

$ac \equiv bd \pmod{g}$  \hspace{1cm} (2)

$a^n \equiv b^n \pmod{h}$  \hspace{1cm} (3)

$10^0 \equiv 1 \pmod{9}$

$10 \equiv 1 \pmod{9}$

$10^n \equiv 1 \pmod{9}$

Let $n$ have digits $a_0, a_1, a_2, \ldots, a_k$

$\therefore n = \sum_{i=0}^{k} a_i \cdot 10^i$

$10^i \equiv 1 \pmod{9}$  \hspace{1cm} (mod 9)

$a_i \equiv a_i \pmod{9}$  \hspace{1cm} (mod 9)

$\Rightarrow a_i \cdot 10^i \equiv a_i \pmod{9}$  \hspace{1cm} (mod 9)

By property (1) \hspace{1cm} $\sum_{i} a_i \cdot 10^i \equiv \sum_{i} a_i \pmod{9}$

$n \equiv \sum_{i} a_i \pmod{9}$

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Grading

50%: Show some general useful knowledge + tinkering.

80%: Some progress.

100%: Generally correct.
Prove by induction for all $n \geq 1$:

$$
\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{(n-1)}{n!} + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}.
$$

\[P(n): \sum_{i=1}^{n} \frac{i}{(i+1)!} = 1 - \frac{1}{(n+1)!}.\]

**Tinker:**

- **P(1):** \( \frac{1}{2!} = 1 - \frac{1}{2!} \) √
- **P(2):** \( \frac{1}{2!} + \frac{2}{3!} = \frac{1}{2} + \frac{2}{6} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6} = 1 - \frac{1}{6} = 1 - \frac{1}{3!} \) √

**Proof by Induction**

**Base case:** $P(1)$ is true.

**Induction step:** Assume $P(n)$: \( \sum_{i=1}^{n} \frac{i}{(i+1)!} = 1 - \frac{1}{(n+1)!} \)

Prove $P(n+1)$: \( \sum_{i=1}^{n+1} \frac{i}{(i+1)!} = 1 - \frac{1}{(n+2)!} \)

\[
\sum_{i=1}^{n+1} \frac{i}{(i+1)!} = \sum_{i=1}^{n} \frac{i}{(i+1)!} + \frac{n+1}{(n+2)!} \\
= 1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!} \quad \text{(induction hypothesis)} \\
= 1 - \frac{1}{(n+1)!} \left[ \frac{(n+2) - (n+1)}{(n+2)!} \right] \\
= 1 - \frac{1}{(n+2)!} \quad \text{as was to be shown.}
\]

By induction $P(n)$ is true for all $n \geq 1$.

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**Grading**

- \(50\%\): Tinkering, understood problem, setup basic induction infrastructure.
- \(80\%\): Some progress, e.g. linked $P(n+1)$ to $P(n)$.
- \(100\%\): Basically correct induction.
$A_n$ satisfies the recurrence below. Find a formula for $A_n$ and prove your answer:

$A_0 = 1, A_1 = 2$ and $A_n = A_{n-1} + 2A_{n-2}$ for $n \geq 2$

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
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<th>4</th>
<th>5</th>
<th>6</th>
<th>...</th>
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<tbody>
<tr>
<td>$A_n$</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>32</td>
<td>64</td>
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\[ \text{Guess } A_n = 2^n \quad m \geq 0. \]

\[ P(0): A_0 = 2^0 \]

**Proof by induction (strong).**

**Base case:**

$A_0 = 1 = 2^0 \quad \checkmark$

$A_1 = 2 = 2^1 \quad \checkmark$

\( \{ P(1) \text{ and } P(2) \} \)

**Induction step:**

Assume: $A_0 = 2^0, A_1 = 2^1, \ldots, A_n = 2^n \quad m \geq 1$

Prove: $A_{n+1} = 2^{n+1}$

By the recursion:

\[ A_{n+1} = A_n + 2 \cdot A_{n-1} \]
\[ = 2^n + 2 \cdot 2^{n-1} \]
\[ = 2^n + 2^n \]
\[ = 2 \cdot 2^n \]
\[ = 2^{n+1} \quad \text{as was to be shown} \]

By induction, $A_n = 2^n$ for all $n \geq 0$.

**Grading**

50%: Tinkers and make right guess

80%: Proof by strong induction framework.

100%: Basically correct induction and linking $A_{n+1}$ to $A_n, A_{n-1}$.
How many subsets of \{1,2,3,4,5,6\} contain some consecutive numbers?

For example the subset \{1,2,4,5\} has consecutive numbers but the subset \{1,3\} does not.

80% if you list all $2^6 = 64$ subsets and count which have consec #.

### Build Up Counting

Let $W(n)$ be the # of subsets with some consecutive #s when the base set has $n$ consec #s.

Let $Q(n)$ be the # of subsets with NO consecutive #s when the base set has $n$ consec #s.

$W(n) = 2^n - Q(n)$

Subsets in $Q(n)$

- $\{1\}$
- $\{2\}$
- $\{3\}$
- $\{4\}$
- $\{5\}$
- $\{6\}$
- $\{1,2\}$
- $\{1,3\}$
- $\{1,4\}$
- $\{1,5\}$
- $\{1,6\}$
- $\{2,3\}$
- $\{2,4\}$
- $\{2,5\}$
- $\{2,6\}$
- $\{3,4\}$
- $\{3,5\}$
- $\{3,6\}$
- $\{4,5\}$
- $\{4,6\}$
- $\{5,6\}$

Q(n) is the total number of subsets of $\{1,2,3,4,5,6\}$ that do not contain any consecutive numbers. To find $Q(n)$, we can use the recurrence relation:

$Q(n) = Q(n-1) + Q(n-2)$

- $Q(0) = 1$
- $Q(1) = 2$
- $Q(2) = 3$
- $Q(3) = 5$
- $Q(4) = 8$
- $Q(5) = 13$
- $Q(6) = 21$

Using this, we find:

$W(6) = 64 - 21 = 43$

$W(n) = 2^6 - Q(6)$

Therefore, the answer is 43.

### Grading

- 50%: Tinker and understand problem
- 30%: Formulate appropriate notation
- 20%: Basically correct reasoning and answer.