Advanced Counting
Reading

  – Chapter 14
Overview

• Sequences with repetition
  – Anagrams

• Inclusion-exclusion: extending the sum-rule to overlapping sets
  – Derangements

• Pigeonhole principle
  – Social twins
  – Subset sums
Selecting $k$ from $n$ Distinguishable Objects

- Last time we saw the number of ways to select $k$ from $n$ objects in the following settings:
  1. $k$-sequence with repetition: $n^k$
  2. $k$-sequence without repetition (permutations): $\frac{n!}{(n-k)!}$
  3. $k$-subset with repetition (candy selection problem): $\binom{n+k-1}{k-1}$
  4. $k$-subset without repetition (combinations): $\frac{n!}{(n-k)!k!}$

- How about a sequences that contain a specific number per object type?
  - E.g., 5 red candies, 4 blue candies, 3 green candies
  - Known as $(k_1, k_2, ..., k_r)$-sequences
Selecting $k$ from $n$ Distinguishable Objects, cont’d

- How about sequences that contain a specific number per object type?
  - E.g., 5 red candies, 4 blue candies, 3 green candies
  - Known as $(k_1, k_2, \ldots, k_r)$-sequences

- How do we count this weird set?
  - Look at all possible positions for each candy type
  - First, count all possible ways to place red candies. How many is that?
    
    \[
    \binom{12}{5}
    \]

  - How many ways can we place the remaining blue candies?
    - We have 7 remaining slots and 4 candies, so \(\binom{7}{4}\)

  - Finally, we have 3 remaining slots and 3 green candies: \(\binom{3}{3}\)
Selecting \( k \) from \( n \) Distinguishable Objects, cont’d

• How about a sequences that contain a specific number per object type?
  – E.g., 5 red candies, 4 blue candies, 3 green candies
  – Known as \((k_1, k_2, \ldots, k_r)\)-sequences

• The final number of ways we can order the candies is:

\[
\binom{12}{5,4,3} = \binom{12}{5} \times \binom{7}{4} \times \binom{3}{3} = \frac{12!}{5! \cdot 7!} \times \frac{7!}{4! \cdot 3!} \times \frac{3!}{0! \cdot 3!} = \frac{12!}{5! \cdot 4! \cdot 3!}
\]
Anagrams: All “Words” Using the Letters AARDVARK

• A sequence of 8 letters: 3 A’s, 2 R’s, 1 D, 1 V, 1 K

• How many sequences is that?
• Number of such sequences is

\[
\binom{8}{3,2,1,1,1} = \frac{8!}{3!2!1!1!1!} = 3360
\]

• **Exercise.** What is the coefficient of \(x^2y^3z^4\) in the expansion of \((x + y + z)^9\)?
  
  — [**Hint: Sequences of length 9 (why?) with 2 x’s, 3 y’s and 4 z’s.**]

Source: wikipedia
Extending the Sum Rule to Overlapping Sets

• What is the size of $|A \cup B|$?

$$|A \cup B| = |A| + |B| - |A \cap B|$$

– Breaks $A \cup B$ into smaller subsets

• Example. How many numbers in 1, ..., 10 are divisible by 2 or 5?

$A = \{\text{numbers divisible by 2}\}$. $|A| = 5$.

– If the sequence contains $n$ numbers, what is the general formula for $|A|$?

• TINKER! Suppose $n$ is even (as above):

$$|A| = \frac{n}{2}$$

• TINKER! Suppose $n$ is odd:

$$|A| = \frac{n - 1}{2}$$

• We can write this using the short-hand notation $|A| = \left\lfloor \frac{10}{2} \right\rfloor$

• The floor $\lfloor x \rfloor$ function returns the largest integer $n \leq x$
Extending the Sum Rule to Overlapping Sets

• What is the size of $|A \cup B|$?

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- Breaks $A \cup B$ into smaller subsets

• Example. How many numbers in 1, ..., 10 are divisible by 2 or 5?

$A = \{\text{numbers divisible by 2}\}$. $|A| = 5$. \left( |A| = \left\lfloor \frac{10}{2} \right\rfloor \right)\\

$B = \{\text{numbers divisible by 5}\}$. $|B| = 2$.

- If the sequence contains $n$ numbers, what is the general formula for $|B|$?

  • TINKER! Suppose $n$ is divisible by 5 (as above):

    $$|B| = \frac{n}{5}$$

  • TINKER! Suppose $n$ is not divisible by 5:

    $$|B| = \frac{n_{5^{-}}}{5}$$

    - Inventing notation: $n_{5^{-}}$ is the largest integer s.t. $5|n_{5^{-}}$ AND $n_{5^{-}} < n$

  • Notice that once again $|B| = \left\lfloor \frac{10}{5} \right\rfloor$

    - when $n$ not divisible by 5, $\frac{n_{5^{-}}}{5} < \frac{n}{5} < \frac{n_{5+}}{5}$
Extending the Sum Rule to Overlapping Sets

• What is the size of $|A \cup B|$?
  
  $|A \cup B| = |A| + |B| - |A \cap B|$  
  
  – Breaks $A \cup B$ into smaller subsets

• **Example.** How many numbers in 1, ..., 10 are divisible by 2 or 5?

  $A = \{\text{numbers divisible by 2}\}$.  
  $|A| = 5$.  
  $\left(|A| = \left\lfloor \frac{10}{2} \right\rfloor \right)$

  $B = \{\text{numbers divisible by 5}\}$.  
  $|B| = 2$.  
  $\left(|B| = \left\lfloor \frac{10}{5} \right\rfloor \right)$

  $A \cap B = \{\text{numbers divisible by 2 AND 5}\}$.  
  $|A \cap B| = 1$.  
  $\left(|A \cap B| = \left\lfloor \frac{10}{\text{lcm}(2,5)} \right\rfloor \right)$

    – (verify that the $\text{lcm}$ is indeed the number we want above!)

  $A \cup B = \{\text{numbers divisible by 2 OR 5}\}$

  
  $|A \cup B| = |A| + |B| - |A \cap B| = 5 + 2 - 1 = 6$
Inclusion-Exclusion

• What about a union of three sets:
  \[ |A_1 \cup A_2 \cup A_3| \]

• Looking at the disjoint sets in the picture, I claim that the formula is:
  \[ |A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| \]
  – Why?
  – Avoid double-counting: start from largest set, then subtract overlap, then add back the overlap of the subtracted, etc.

• Proof sketch. Consider \( x \in A_2 \cap A_3 \). How many times is \( x \) counted?
  \[ |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| \]
  \[ 0 + 1 + 1 - 0 - 0 - 1 + 0 \]
  – Contribution of \( x \) to sum is +1. Repeat for each region.
  – Should be true for each region. Means there’s no double-counting.
Inclusion-Exclusion, cont’d

• **Example.** Give 3 coats to 3 people so that no one gets their coat. How many ways?
  – How do we split the sets?
    \[ A_i = \{\text{person } i \text{ gets their coat}\}, |A_i| = 2! \]
  – Why? (position \( i \) is fixed)
    \[ A_{ij} = \{\text{people } i \text{ and } j \text{ get their coats}\}, |A_{ij}| = 1! \]
  – Why? (positions \( i \) and \( j \) are fixed)
    \[ A_{123} = \{\text{people } 1, 2 \text{ and } 3 \text{ get their coats}\}, |A_{123}| = 1 \]
  – All positions are fixed
    \[
    |A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_{12}| - |A_{13}| - |A_{23}| + |A_{123}|
    = 2 + 2 + 2 - 1 - 1 - 1 + 1 = 4
    \]
  – The answer we seek is \( 3! - 4 = 2 \)
    – Why?
    – How big is the set of all possible coat assignments?
      \[ 3 \times 2 \times 1 = 3! \]
    – Subtract from those the set \( A = \{\text{at least one person has their coat on}\} \)
      \[ A = A_1 \cup A_2 \cup A_3 \]
  • **Exercise.** How many numbers in \( 1, \ldots, 100 \) are divisible by 2,3 or 5?
• What about the general formula:

\[ |A_1 + A_2 + \cdots + A_n| \]

• It seems that

\[
|A_1 + A_2 + \cdots + A_n| =
\]

\[
= (|A_1| + \cdots + |A_n|)
\]

\[
- (|A_1 \cap A_2| + |A_1 \cap A_3| + \cdots + |A_{n-1} \cap A_n|)
\]

\[
+ (|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_3 \cap A_4| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n|)
\]

\[- \cdots \]

• Claim:

\[ |A_1 + A_2 + \cdots + A_n| = \sum_{k=1}^{n} (-1)^{k+1} \times [\text{sum of all } k\text{-way intersection sizes}] \]

• Proof sketch. Suppose \( x \) lies in \( r \) sets.

  – How many 1-way intersections contain \( x \)?

\[
\binom{r}{1}
\]

  – How many 2-way intersections contain \( x \)?

\[
\binom{r}{2}
\]

  – Verify that \( x \) contributes only 1 to the full sum.
Pigeonhole Principle

• If you have more guests than spare rooms, then some guests will have to share

  – More pigeons than pigeonholes

• Theorem. A pigeonhole has two or more pigeons (if there are more pigeons than pigeonholes).

• Proof. (By contraposition). Suppose no pigeonhole has 2 or more pigeons.
  – Let \( x_i \) be the number of pigeons in hole \( i \), \( x_i \leq 1 \).
  
  \[
  \text{number of pigeons} = \sum_i x_i \leq \sum_i 1 = \text{number of pigeonholes}
  \]
Every Graph Has at Least One Pair of Social Twins

• Two nodes are *social twins* if they have the same degree.
• Consider a connected graph
• What are the nodes’ degrees?

<table>
<thead>
<tr>
<th>v_1</th>
<th>v_2</th>
<th>v_3</th>
<th>v_4</th>
<th>v_5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

- What if we had a graph with 0-degree nodes?
  • Exclude that node and only consider the connected sub-graph

• Degrees 1, 2, ..., (n – 1), are the pigeonholes
• Vertices v_1, v_2, ..., v_n, are the pigeons
• There are n pigeons and (n – 1) pigeonholes, so at least two vertices are in the same degree-bin
• This proof is not very satisfactory (why?)
  - Who are those social twins? What are their degrees?
  - Known as a non-constructive proof
Non-constructive proofs are not always desirable

- A non-constructive proof that P = NP is almost useless
- Sure, we know that they are equal, but that still doesn’t tell us how to factor numbers

Sometimes non-constructive proofs can be valuable

Password checking is a type of non-constructive proof

- You enter your password and you get a “yes/no” answer
- A “no” answer doesn’t leave you any the wiser as to the true password
• Can you find the cat in this image?
Non-Constructive Proof and the Eye-Spy Dilemma, cont’d

• How do I convince you that the cat is in the image without pointing to the cat?
  – I want you to know the problem is fair without revealing the solution
• If only we had an infinite black cloth that has a cat-shaped hole
  – I could slide the image under the cloth until the cat shows up
• Suppose my cloth is not perfect and it reveals a bit more than necessary
Non-Constructive Proof and the Eye-Spy Dilemma, cont’d

- Can you now find the cat in this image?
Zero-Knowledge Proof and the Ali Baba cave

- Suppose that Peggy found a secret word used to open a door in a cave.
- Victor wants to know if Peggy really knows the secret word.
  - (Peggy won’t actually say the word because it’s secret)
- So Victor designs an experiment:
  - Peggy goes in the cave.
  - Victor can’t see which path she takes.
  - Victor flips a coin.
  - If coin comes up heads, Victor asks Peggy to come back using path A (o.w. using B).
  - If Peggy knows the word, she can use any path.
  - If Peggy doesn’t know the word, she has to go back the way she entered.
    - She has a 50% chance of taking the right path.
    - If they repeat this many times, her chance of guessing right goes down to 0.
Subset Sums

• Suppose I pick 10 numbers between 1 and 100
  – Call my set $S$
    • e.g., $S = \{1,2,3,4,5,6,7,8,9,99\}$
  – I claim that at least two distinct subsets of $S$ have the same subset-sum.
  – In my case, this is obvious: $\{1,2\}$ and $\{3\}$
    • This is a constructive proof, but let’s look at a zero-knowledge one also

• A subset’s sum is $x_1 + x_2 + \cdots + x_{10} \leq 10 \times 100 = 1000$

• Pigeonholes: bins corresponding to each possible subset-sum, 1, 2, ..., 1000

• Pigeons: the non-empty subsets of a 10-element set:
  \[2^{10} - 1 = 1023\]

• At least two subsets must be in the same subset-sum-bin.

• Practice. Exercise 14.6.

• Practice. Exercise 14.7.