Strong Induction: Strengthening Induction
  – Chapter 6
• Solving harder problems with induction
  – Proving $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$
• Strengthening the induction hypothesis
  – Proving $n^2 < 2^n$
  – $L$-tiling
• Many flavors of induction
  – Leaping Induction
    • Postage
    • $n^3 < 2^n$
  – Strong induction
    • Fundamental Theorem of Arithmetic
    • Games of Strategy
A Hard Problem: $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$

- **Proof.** $P(n)$: $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$

1. **[Base case]** $P(1)$ claims that $1 \leq 2$, which is T

2. **[Induction step]** Show $P(n) \rightarrow P(n + 1)$ for all $n \geq 1$. Direct proof.
   - Assume (induction hypothesis) $P(n)$ is T: $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$
   - Show $P(n + 1)$ is T:
     $$\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} \leq 2\sqrt{n + 1}$$
     $$\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} = \sum_{i=1}^{n} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n+1}}$$
     $$\leq 2\sqrt{n} + \frac{1}{\sqrt{n+1}}$$
     **[key step]**
     $$\leq 2\sqrt{n + 1}$$
     **[induction hypothesis]**

   - Hm, now what??

   - **Lemma:** $2\sqrt{n} + \frac{1}{\sqrt{n+1}} \leq 2\sqrt{n + 1}$
Lemma: \( 2\sqrt{n} + \frac{1}{\sqrt{n+1}} \leq 2\sqrt{n} + 1 \)

- Proof. By contradiction.
  - Assume
    \[
    2\sqrt{n} + \frac{1}{\sqrt{n+1}} > 2\sqrt{n} + 1
    \]
  - It follows that (by multiplying by \( \sqrt{n+1} \))
    \[
    2\sqrt{n(n+1)} + 1 > 2(n+1)
    \]
    \[
    2\sqrt{n(n+1)} > 2n + 1
    \]
    \[
    4n(n + 1) > (2n + 1)^2
    \]
    \[
    4n^2 + 4n > 4n^2 + 4n + 1
    \]
    \[
    0 > 1
    \]
  - Contradiction!
A Hard Problem: $\Sigma_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$

- Proof. $P(n): \Sigma_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$

1. **[Base case]** $P(1)$ claims that $1 \leq 2$, which is T

2. **[Induction step]** Show $P(n) \rightarrow P(n + 1)$ for all $n \geq 1$. Direct proof.
   - Assume (induction hypothesis) $P(n)$ is T: $\Sigma_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$
   - Show $P(n + 1)$ is T: $\Sigma_{i=1}^{n+1} \frac{1}{\sqrt{i}} \leq 2\sqrt{n + 1}$
     
     $\Sigma_{i=1}^{n+1} \frac{1}{\sqrt{i}} = \Sigma_{i=1}^{n} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n+1}}$ [key step]
     
     $\leq 2\sqrt{n} + \frac{1}{\sqrt{n+1}}$ [induction hypothesis]
     
     $\leq 2\sqrt{n + 1}$ [Lemma]
   - So, $P(n) \rightarrow P(n + 1)$

3. By induction, $P(n)$ is T $\forall n \geq 1$. 


Proving Stronger Claims

- Prove that \( n^2 \leq 2^n \) for \( n \geq 4 \)

**Proof attempt.** [By induction]

- **[Base case]** \( P(4) \) claims that \( 16 \leq 16 \), which is T

- **[Induction step]** Assume \( P(n) \) is T: \( n^2 \leq 2^n \) for \( n \geq 4 \)
  - Need to show \( P(n) \rightarrow P(n+1): \)
    \[ (n + 1)^2 \leq 2^{n+1} \]
  - Note that \( (n + 1)^2 = n^2 + 2n + 1 \leq 2^n + 2n + 1 \)
  - If only we could show \( 2n + 1 \leq 2^n \)
    - Then \( 2^n + 2n + 1 \leq 2^n + 2^n = 2^{n+1} \)
  - With induction, it can be easier to prove a stronger claim.
Strengthen the claim: \( Q(n) \) Implies \( P(n) \)

- Consider a new claim \( Q(n): (i) \) \( n^2 \leq 2^n \) AND \( (ii) \) \( 2n + 1 \leq 2^n \)

- **Proof.** [By induction]
  1. **[Base case]** \( Q(4) \) claims \( 16 \leq 16 \) AND \( 9 \leq 16 \); both are T
  2. **[Induction step]** Show \( Q(n) \rightarrow Q(n + 1) \) for \( n \geq 4 \). Direct proof
     - Assume \( Q(n) \) is T: \( (i) \) \( n^2 \leq 2^n \) AND \( (ii) \) \( 2n + 1 \leq 2^n \)
     - Show \( Q(n + 1) \) is T:
       \[(i) \quad (n + 1)^2 \leq 2^{(n+1)} \text{ AND } (ii) \quad 2(n + 1) + 1 \leq 2^{(n+1)}\]
       \[(i): \quad (n + 1)^2 = n^2 + 2n + 1 \]
       \[
       \leq 2^n + 2n + 1 \leq 2^n + 2^n = 2^{n+1}
       \]
       - (From the induction hypothesis: \( n^2 \leq 2^n \) AND \( 2n + 1 \leq 2^n \))
       \[(ii): \quad 2(n + 1) + 1 = 2 + 2n + 1 \]
       \[
       \leq 2^n + 2^n = 2^{n+1}
       \]
       - (Because \( 2 \leq 2^n \) and \( 2n + 1 \leq 2^n \) from the induction hypothesis)
     - So \( Q(n + 1) \) is T
  3. By induction, \( Q(n) \) is T for \( n \geq 4 \)
**L-Tile Land**

- Can you tile a $2^n \times 2^n$ patio missing a center square (there’s a pot there!). You only have $L$-shaped tiles

- **TINKER!**
  - when $n = 1$
  - when $n = 2$
  - when $n = 3$

- $P(n)$: The $2^n \times 2^n$ grid minus a center-square can be $L$-tiled.
**L-Tile Land: Induction Idea**

- Suppose $P(n)$ is true. What about $P(n + 1)$?
- The $2^{n+1} \times 2^{n+1}$ patio can be decomposed into four $2^n \times 2^n$ patios

- **Problem.** Corner squares are missing. $P(n)$ can be used only if center-square is missing.
- **Solution.** Strengthen claim to also include patios missing corner-squares. $Q(n)$:
  - (i) The $2^n \times 2^n$ grid missing a **center-square** can be $L$-tiled; AND
  - (ii) The $2^n \times 2^n$ grid missing a **corner-square** can be $L$-tiled
**L-Tile Land: Induction Proof of Stronger Claim**

- Assume $Q(n)$:
  - (i) The $2^n \times 2^n$ grid missing a **center-square** can be $L$-tiled; and
  - (ii) The $2^n \times 2^n$ grid missing a **corner-square** can be $L$-tiled

- Induction step: Must prove two things for $Q(n + 1)$, namely (i) and (ii).
  - (i) Center square missing
  - (ii) Corner square missing

- Exercise: Add base cases and complete the formal proof.

- **Exercise 6.4.** What if the missing square is some random square?
  - Strengthen further.
• Prove $P(n): n^3 < 2^n$, for all $n \geq 10$

• *Proof attempt.* [By induction]
  – *[Base case]* $P(10)$ claims $1000 = 10^3 < 2^{10} = 1024$.
    • True.
  – *[Induction step]* Assume $P(n)$ is T: $n^3 < 2^n$ for $n \geq 10$.
    • Need to show $P(n + 1)$ is T:
      \[(n + 1)^3 < 2^{n+1}\]
      – Seems hard
  • Consider $P(n + 2): (n + 2)^3 < 2^{n+2}$?
    \[(n + 2)^3 = n^3 + 6n^2 + 12n + 8\]
    \[< n^3 + n \cdot n^2 + n^2 \cdot n + n^3\]
    » (Because $n \geq 10 \rightarrow 6 < n, 12 < n^2, 8 < n^3$)
    \[(n + 2)^3 < n^3 + n \cdot n^2 + n^2 \cdot n + n^3 = 4n^3\]
    \[< 4 \cdot 2^n = 2^{n+2}\]
    » (From induction hypothesis: $P(n): n^3 < 2^n$)
    – i.e., $P(n) \rightarrow P(n + 2)$
• Not quite induction yet. What can we do?
A Tricky Induction Problem, cont’d

• Prove \( P(n) \): \( n^3 < 2^n \), for all \( n \geq 10 \)

• *Proof. [By induction]*

1. **[Base cases]** \( P(10) \) claims \( 1000 = 10^3 < 2^{10} = 1024 \).
   \( P(11) \) claims \( 1331 = 11^3 < 2^{11} = 2048 \).
   – Both are T.

2. **[Induction step]** Assume \( P(n) \) is T: \( n^3 < 2^n \) for \( n \geq 10 \).
   – Need to show \( P(n) \) \( \rightarrow \) \( P(n + 2) \): \( (n + 2)^3 < 2^{n+2} \)
      
      • Consider \( P(n + 2) \): \( (n + 2)^3 < 2^{n+2} \)?
    \[
    (n + 2)^3 = n^3 + 6n^2 + 12n + 8 \\
    < n^3 + n \cdot n^2 + n^2 \cdot n + n^3
    
    » (Because \( n \geq 10 \rightarrow 6 < n, 12 < n^2, 8 < n^3 \))
    \[
    (n + 2)^3 < n^3 + n \cdot n^2 + n^2 \cdot n + n^3 = 4n^3 < 4 \cdot 2^n = 2^{n+2}
    
    » (From induction hypothesis: \( P(n) \): \( n^3 < 2^n \))
      – i.e., \( P(n) \) \( \rightarrow \) \( P(n + 2) \)

3. By induction, \( P(n + 2) \) is T for all \( n \geq 10 \)
   – Already showed \( P(10) \) and \( P(11) \) are T.
Leaping Induction

• **Induction.** One base case.
  \[ P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow \cdots \]

• **Leaping Induction.** More than one base case.
  \[ P(1) \rightarrow P(3) \rightarrow P(5) \rightarrow \cdots \]
  \[ P(2) \rightarrow P(4) \rightarrow P(6) \rightarrow \cdots \]

• **Example.** Postage greater than 5¢ can be made using 3¢ and 4¢ stamps.

<table>
<thead>
<tr>
<th>3¢</th>
<th>4¢</th>
<th>5¢</th>
<th>6¢</th>
<th>7¢</th>
<th>8¢</th>
<th>9¢</th>
<th>10¢</th>
<th>11¢</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
<td>-</td>
<td>3,3</td>
<td>3,4</td>
<td>4,4</td>
<td>3,3,3</td>
<td>3,4,3</td>
<td>4,4,3</td>
</tr>
</tbody>
</table>

• \( P(n) \): Postage of \( n \text{¢} \) can be made using only 3¢ and 4¢ stamps.
  \[ P(n) \rightarrow P(n + 3) \text{ (add a 3¢ stamp to } n) \]

• **Practice.** Exercise 6.6
Fundamental Theorem of Arithmetic

• The fundamental theorem of arithmetic states that
  \[ 2024 = 2 \times 2 \times 2 \times 11 \times 23 \]
  
  – Huh?
  
  – Well, it says more than that 😊

• Theorem [The primes (\(\mathcal{P} = \{2,3,5,7,11,13,\ldots\}\)) are the atom numbers]. Suppose \(n \geq 2\) is natural number. Then:
  
  – (i) \(n\) can be written as a product of factors all of which are prime.
  
  – (ii) The representation of \(n\) as a product of primes is unique (up to reordering).

• What is \(P(n)\)?

  \[ P(n): n \text{ is a product of primes} \]

• What is the first thing we do?
  
  – TINKER!
Fundamental Theorem of Arithmetic

- The prime-factor decomposition of 2024 is:
  \[ 2024 = 2 \times 2 \times 2 \times 11 \times 23 \]

- **Theorem.** [The primes \( \mathcal{P} = \{2,3,5,7,11,13, \ldots \} \) are the atom numbers]. Suppose \( n \geq 2 \). Then:
  - (i) \( n \) **can be written as a product of factors all of which are prime.**
  - (ii) The representation of \( n \) as a product of primes is unique (up to reordering).

- What is \( P(n) \)?
  \[ P(n): n \text{ is a product of primes} \]

- What is the prime-factor decomposition of 2025:
  \[ 2025 = 5 \times 5 \times 3 \times 3 \times 3 \times 3 \]

- Wow! No similarity between the factors of 2024 and 2025
  - **How will** \( P(n) \) **help us to prove** \( P(n + 1) \)?
Much “Stronger” Induction Claim

- Do smaller values of \( n \) help with 2025?
  - Yes, \( 2025 = 25 \times 81 \)
    \[
P(25) \land P(81) \rightarrow P(2025)
    \]
  - (like leaping induction)

- **Much Stronger Claim:**
  - \( Q(n) \): 2, 3, ..., \( n \) are all products of primes.
  - Compare with: \( P(n) \): \( n \) is a product of primes
    \[
    Q(n) = P(2) \land P(3) \land P(4) \land \cdots \land P(n)
    \]

- **Surprise!** The much stronger claim is *much* easier to prove.
  - Also, \( Q(n) \rightarrow P(n) \)
Fundamental Theorem of Arithmetic: Proof of \((i)\)

- Recall \(P(n): n\) is product of primes.
  - Recall \(Q(n) = P(2) \land P(3) \land \cdots \land P(n)\)

- **Proof.** [By induction that \(Q(n)\) is \(T\) for all \(n \geq 2\).]

1. **[Base case].** \(Q(1)\) claims that 2 is product of primes. True.

2. **[Induction step]** Show that \(Q(n) \rightarrow Q(n + 1)\) for \(n \geq 2\). Direct proof.
   - Assume \(Q(n)\) is \(T\): each of 2, 3, ..., \(n\) are products of primes
   - Show \(Q(n + 1)\) is \(T\): each of 2, 3, ..., \(n\), \(n + 1\) are products of primes
   - Since we assumed \(Q(n)\), we know 2, 3, ..., \(n\) are products of primes
   - To prove \(Q(n + 1)\), we only need to prove \(n + 1\) is a product of primes!
Fundamental Theorem of Arithmetic: Proof of \((i)\)

• **Proof.** [By induction that \(Q(n)\) is T for all \(n \geq 2\).]

1. **[Base case]**. \(Q(1)\) claims that 2 is product of primes. True.

2. **[Induction step]** Show that \(Q(n) \rightarrow Q(n + 1)\) for \(n \geq 2\). Direct proof.
   - Assume \(Q(n)\) is T: each of 2, 3, \(\ldots\), \(n\) are products of primes
   - Show \(Q(n + 1)\) is T: each of 2, 3, \(\ldots\), \(n\), \(n + 1\) are products of primes
   - Since we assumed \(Q(n)\), we know 2, 3, \(\ldots\), \(n\) are products of primes
   - To prove \(Q(n + 1)\), we only need to prove \(n + 1\) is a product of primes!
     - Case 1: \(n + 1\) is prime.
       - Done, nothing to prove.
     - Case 2: \(n + 1\) is not prime,
       - i.e., \(n + 1 = kl\), where \(2 \leq k, l \leq n\).
       - What now?
         » Use induction hypothesis!
         \[P(k): k \text{ is product of primes}; P(l): l \text{ is product of primes.}\]
         - i.e., \(n + 1 = kl\) is a product of primes and \(Q(n + 1)\) is T

3. By induction, \(Q(n)\) is T, \(\forall n \geq 2\).
Strong Induction

• **Strong Induction.** To prove \( P(n) \) \( \forall n \geq 1 \) by strong induction, you use induction to prove the *stronger* claim:
  
  – \( Q(n) \): each of \( P(1), P(2), \ldots, P(n) \) are T

• Ordinary induction
  
  – Base case: Prove \( P(1) \)
  – Induction step: Assume \( P(n) \) and prove \( P(n + 1) \)

• **Strong induction**
  
  – Base case: Prove \( Q(1) = P(1) \)
  – Induction step: Assume \( Q(n) = P(1) \land P(2) \land P(3) \land \cdots \land P(n) \) and prove \( P(n + 1) \)

• **Strong induction is always easier**
Every $n \geq 1$ has a binary expansion

- What is $P(n)$ more precisely?
  - $P(n)$: Every $n \geq 1$ is a sum of distinct powers of 2 (its binary expansion)
  - E.g., what is the binary expansion of 22?

\[
22 = 2^4 + 2^2 + 2^1 \quad (22_{binary} = 10110)
\]
Every \( n \geq 1 \) has a binary expansion

- **Proof Sketch.**
- **[Base case]** \( P(1) \) is T: \( 1 = 2^0 \)
- **[Induction step]** Assume \( P(1) \land P(2) \land \cdots \land P(n) \) and prove \( P(n + 1) \)
  - If \( n \) is even, then
    \[ n + 1 = 2^0 + n_{\text{binary}} \]
    - e.g., \( 23 = 2^4 + 2^2 + 2^1 + 2^0 \)
  - If \( n \) is odd, then multiply each term in the expansion of \( \frac{1}{2}(n + 1) \) by 2
    - This gets us \( n + 1 \)
    - e.g., \( 24 = 2 \times 12_{\text{binary}} = 2 \times (2^3 + 2^2) = 2^4 + 2^3 \)
    - Why does \( \frac{1}{2}(n + 1) \) have an expansion?
      - Strong induction!
- **Exercise.** Give the formal proof by strong induction.
Applications of Induction

• Greedy or recursive algorithms, games of strategy
• Consider the game of Equal Pile Nim (old English/German: to steal or pilfer)
  – two players take turns taking pennies from two equal rows of pennies
  – each player can take an arbitrary number of pennies from one row
  – the player to take the last stone wins

• Claim: $P(n)$: Player 2 can win the game that starts with $n$ pennies per row.
  – Equalization strategy:

  • Player 2 can always return the game to smaller equal piles.
  • If Player 2 wins the smaller game, Player 2 wins the larger game.
    • That’s strong induction!

• Exercise. Give the full formal proof by strong induction.
• Challenge. What about more than 2 piles? What about unequal piles? (Problem 6.20).
Investigate Further in the Problems

• Uniqueness of binary representation as a sum of distinct powers of 2:
  – Problem 6.27

• General Nim:
  – Problem 6.39
Checklist When Approaching an Induction Problem

• Are you trying to prove a “For all . . .” claim?

• Identify the claim $P(n)$, especially the parameter $n$. Here is an example.
  – Prove: geometric mean $\leq$ arithmetic mean. What is $P(n)$? What is $n$?
  – $P(n)$: geometric mean $\leq$ arithmetic mean for every set of $n > 0$ numbers
    – **Identifying the right claim is important.**
      You may fail because you try to prove too much. Your $P(n + 1)$ is too heavy a burden. You may fail
      because you try to prove too little. Your $P(n)$ is too weak a support. You must balance the strength of
      your claim so that the support is just enough for the burden. —G. Polya (paraphrased).

• Tinker. Does the claim hold for small $n$ ($n = 1, 2, 3, ...$)? These become base cases.

• Tinker. Can you see why (say) $P(5)$ follows from $P(1), P(2), P(3), P(4)$?
  – This is the crux of induction; to build up from smaller $n$ to a larger $n$.

• Determine the type of induction: try strong induction first.

• Write out the skeleton of the proof to see exactly what you need to prove.

• Determine and prove the base cases.

• Prove $P(n + 1)$ in the induction step. You must use the induction hypothesis.