Midterm

110 Minutes

First Name: ____________________________

Last Name: ____________________________

RIN: ____________________________

NO COLLABORATION or electronic devices.
Any violations will result in an F.
No questions allowed during the test unless you think there is a mistake.

GOOD LUCK!

10 points per correct multiple-choice answer. Circle exactly one answer.
20 points per correct answer to Problems 2-6.
For [**Show Work**] problems, you **MUST** show correct work to get credit.
Correct answers with no explanation will get a 0.

1	2	3	4	5	6	Total
150	20	20	20	20	20	250

1. Suppose a function $S(n) \in \Theta(n^2)$. What do we know?

A $S(n) \in \Theta(n)$
B $S(n) \in O(n)$
C $S(n) \in O(n^2)$
D $S(n) \in \Theta(n^3)$
E None of the above.

2. What is the asymptotic behavior of the sum $S(n) = \sum_{i=1}^{n} i^6$?

A $\Theta(n^6)$
B $\Theta(n^7)$
C $\Theta(n^8)$
D $\Theta(n^9)$
E None of the above.

3. [**Show Work**] What is the value of the sum $S = \sum_{i=1}^{5} \sum_{j=1}^{5} ij$?

A 200
B 225
C 250
D 275
E 300

4. [**Show Work**] What is the asymptotic behavior of the sum $S(n) = \sum_{i=1}^{n} \sum_{j=1}^{i} j$?

A $\Theta(n^4)$
B $\Theta(n^5)$
C $\Theta(n^6)$
D $\Theta(n^7)$
E None of the above.

5. [**Show Work**] We know that $gcd(13, 11) = 13x + 11y$. Which of the following values are possible for x and y?

A $x = 4, y = -5$
B $x = -4, y = 5$
C $x = 5, y = -6$
D $x = -5, y = 6$
E None of the above.

6. What is the remainder when 14^{100} is divided by 15?

   A -2
   B -1
   C 0
   D 1
   E 2

7. Consider a graph G with degree sequence $[3,3,3,3,3]$. How many edges does G have?

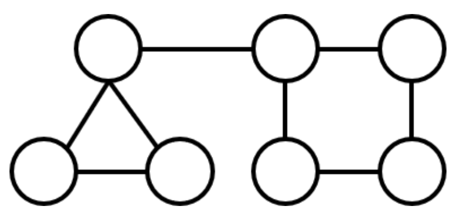
   A 4
   B 6
   C 8
   D 10
   E None of the above.

8. Consider a graph G with degree sequence $[4,4,4,4,4]$. How many edges does G have?

   A 8
   B 9
   C 10
   D 11
   E None of the above.

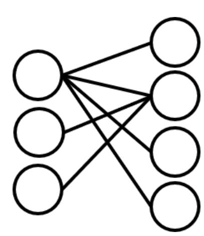
9. Consider the graph below. What is the minimum number of colors needed to color this graph so that no two neighbors get the same color, i.e., what is the graph's chromatic number?

   A 1
   B 2
   C 3
   D 4
   E None of the above.



10. Consider the bipartite graph below. Is a left matching possible?

    A Yes.
    B No.
    C The graph is not bipartite.
    D Yes, because Hall's Theorem applies.
    E None of the above.



11. [**Show Work**] How many subsets of $\{1, 2, 3, 4, 5, 6\}$ contain at least three even numbers?

- A 2^3
- B 2^4
- C 2^5
- D 2^6
- E None of the above.

12. Consider all 7-bit binary strings with a 1 in the second position and a 0 in the third position? How many such strings are there?

- A 16
- B 32
- C 64
- D 128
- E None of the above.

13. When making the exam, I initially came up with 20 multiple-choice questions. Now I need to shrink that to 15 questions. How many sets of 15 questions (ignoring different orderings) are there?

- A $\binom{15}{20}$
- B $20 \times 19 \times 18 \times 17 \times 16$
- C 200
- D 250
- E None of the above.

14. In how many ways can you misspell SPRING, assuming you use the same letters?

- A $6!$
- B 2^6
- C $2^6 - 1$
- D $6! - 1$
- E None of the above.

15. [**Show Work**] What is the last digit of 13^5?

- A 0
- B 1
- C 2
- D 3
- E None of the above.

Problem 2. Prove using contradiction: $\forall k \in \mathbb{N}, \sqrt{k} + \sqrt{k+1} < \sqrt{4k+2}$**.**

Problem 3. Prove using induction: $\forall n \geq 1 : 1 + 2 + \cdots + n \leq n^2$**.**

[Note: you are not allowed to use the identity $1 + 2 + \cdots + n = \frac{n}{2}(n + 1)$.]

Problem 4. Recall the rooted binary tree (RBT) recursive definition:

① The empty tree ε is an RBT. [**base case**]

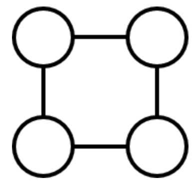
② If T_1 and T_2 are RBTs with roots r_1 and r_2, then linking r_1 and r_2 to a new root r gives a new RBT with root r. [**constructor**]

③ No other tree is an RBT. [**minimality**]

Prove using structural induction that the degree of the root in any rooted binary tree is at most 2.

Problem 5. The diameter of a graph is the distance between the two vertices that are furthest apart. Recall that C_n denotes a cycle graph with n vertices (C_4 is shown below). What is the diameter of C_n?

[You don't need to prove your answer but you need to provide sufficient reasoning to justify the formula.]



Problem 6. A graph is r-regular if every vertex has the same degree r. Suppose r is even. Show that for all even r and for all n such that $n > r$, there exists an r-regular graph with n vertices. Tinker, tinker, tinker.

Scratch

Scratch