# Midterm

110 Minutes

First Name: SOLUTIONS

Last Name: ______________________

RIN: ______________________

**NO COLLABORATION or electronic devices.**
**Any violations will result in an F.**
**No questions** allowed during the test unless you think there is a mistake.

## GOOD LUCK!

**10 points** per correct multiple-choice answer. Circle exactly one answer.
**20 points** per correct answer to Problems 2-6.
For [**Show Work**] problems, you **MUST** show correct work to get credit.
Correct answers with no explanation will get a 0.

| 1 | 2 | 3 | 4 | 5 | 6 | Total |
|---|---|---|---|---|---|---|
| | | | | | | |
| 150 | 20 | 20 | 20 | 20 | 20 | 250 |

$f(n) \in \Theta(n^2) \Rightarrow f(n) \text{ "=" } n^2$

1. Suppose a function $S(n) \in \Theta(n^2)$. What do we know?

   A $S(n) \in \Theta(n)$
   B $S(n) \in O(n)$
   C $S(n) \in O(n^2)$
   D $S(n) \in \Theta(n^3)$
   E None of the above.

$f(n) \in O(n^2) \Rightarrow f(n) \text{ "}\leq\text{" } n^2$

So $\Theta \subseteq O$

2. What is the asymptotic behavior of the sum $S(n) = \sum_{i=1}^{n} i^6$?

   A $\Theta(n^6)$
   B $\Theta(n^7)$
   C $\Theta(n^8)$
   D $\Theta(n^9)$
   E None of the above.

Using intergration method

$$\sum_{i=1}^{n} i^6 \approx \int_0^n x^6 \, dx = \frac{x^7}{7}\Big|_0^n = \frac{n^7}{7} \in \Theta(n^7)$$

3. [**Show Work**] What is the value of the sum $S = \sum_{i=1}^{5}\sum_{j=1}^{5} ij$?

   A 200
   B 225
   C 250
   D 275
   E 300

$1+2+3+4+5 = 15$

Constant rule! $S = \sum_{i=1}^{5} i \sum_{j=1}^{5} j$

$= 15 \times 15 = 225$

4. [**Show Work**] What is the asymptotic behavior of the sum $S(n) = \sum_{i=1}^{n}\sum_{j=1}^{i} j$?

   A $\Theta(n^4)$
   B $\Theta(n^5)$
   C $\Theta(n^6)$
   D $\Theta(n^7)$
   E None of the above.

Start from innermost sum:

$\sum_{j=1}^{i} j = i \cdot (i+1) \div 2 \approx i^2$ (ignore low-order terms)

$\sum_{i=1}^{n} i^2 = \frac{1}{6} n(n+1)(2n+1) \in \Theta(n^3)$

5. [**Show Work**] We know that $gcd(13, 11) = 13x + 11y$. Which of the following values are possible for $x$ and $y$?

   A $x = 4, y = -5$
   B $x = -4, y = 5$
   C $x = 5, y = -6$
   D $x = -5, y = 6$
   E None of the above.

$gcd(13, 11) = gcd(11, 2)$ $[2 = 13 - 11]$

$= gcd(2, 1)$ $[1 = 11 - 5 \times 2$

$= 6 \times 11 - 5 \times 13$

$= 1$

$\boxed{x = -5 \quad y = 6}$

6. What is the remainder when $14^{100}$ is divided by 15?

- A -2
- B -1
- C 0
- D 1
- E 2

$14 \equiv -1 \pmod{15}$

$14^{100} \equiv (-1)^{100} \equiv 1 \pmod{15}$

7. Consider a graph $G$ with degree sequence $[3, 3, 3, 3, 3]$. How many edges does $G$ have?

- A 4
- B 6
- C 8
- D 10
- E None of the above.

Sum is 15 : odd!

$\sum_i \delta_i = 2|E|$ : even!

Graph doesn't exist!

8. Consider a graph $G$ with degree sequence $[4, 4, 4, 4, 4]$. How many edges does $G$ have?

- A 8
- B 9
- C 10
- D 11
- E None of the above.

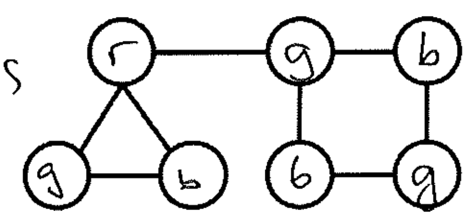
$20 = \sum_i \delta_i = 2|E|$

$|E| = 10$

9. Consider the graph below. What is the minimum number of colors needed to color this graph so that no two neighbors get the same color, i.e., what is the graph's chromatic number?

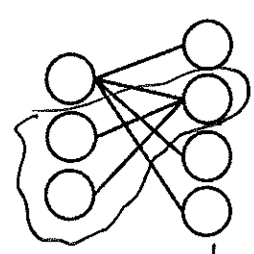
- A 1
- B 2
- C 3
- D 4
- E None of the above.

largest clique has size 3




10. Consider the bipartite graph below. Is a left matching possible?

- A Yes.
- B No.
- C The graph is not bipartite.
- D Yes, because Hall's Theorem applies.
- E None of the above.



$|X| > |N(X)|$

11. **[Show Work]** How many subsets of $\{1, 2, 3, 4, 5, 6\}$ contain at least three even numbers?

- A $2^3$
- B $2^4$
- C $2^5$
- D $2^6$
- E None of the above.

All subsets contain all even numbers. Need all subsets of the (3) odd numbers: $2^3$.

12. Consider all 7-bit binary strings with a 1 in the second position and a 0 in the third position? How many such strings are there?

- A 16
- B 32
- C 64
- D 128
- E None of the above.

_ 1 0 _ _ _ _

Same as all 5-bit strings

$2^5 = 32$

13. When making the exam, I initially came up with 20 multiple-choice questions. Now I need to shrink that to 15 questions. How many sets of 15 questions (ignoring different orderings) are there?

- ~~A~~ $\binom{15}{20}$
- ~~B~~ $20 \times 19 \times 18 \times 17 \times 16$ permutations
- ~~C~~ 200
- ~~D~~ 250
- E None of the above.

$\binom{20}{15}$

14. In how many ways can you misspell SPRING, assuming you use the same letters?

- ~~A~~ 6! permutations including SPRING
- B $2^6$
- C $2^6 - 1$
- D $6! - 1$ permutations excluding SPRING
- E None of the above.

15. **[Show Work]** What is the last digit of $13^5$?

- A 0
- B 1
- C 2
- D 3
- E None of the above.

$13 \equiv 3 \pmod{10}$

$13^5 \equiv 3^5 \pmod{10}$

$\equiv 3^2 \cdot 3^3 \pmod{10}$

$\equiv -1 \cdot 7 \equiv -7 \equiv 3 \pmod{10}$

**Problem 2. Prove using contradiction:** $\forall k \in \mathbb{N}, \sqrt{k} + \sqrt{k+1} < \sqrt{4k+2}$.

Assume for a contradiction that

$$\sqrt{k} + \sqrt{k+1} \geq \sqrt{4k+2}$$

Progress (50 %)

Raise both sides to 2nd power:

$$k + 2\sqrt{k(k+1)} + k+1 \geq 4k+2$$

$$2\sqrt{k(k+1)} \geq 2k + 1$$

80 %

Raise to 2nd power again:

$$\cancel{4k^2} + \cancel{4k} \geq \cancel{4k^2} + \cancel{4k} + 1$$

$$0 \geq 1.$$

100 %

FISHY!!

**Problem 3. Prove using induction:** $\forall n \geq 1 : 1 + 2 + \cdots + n \leq n^2$.

[Note: you are not allowed to use the identity $1 + 2 + \cdots + n = \frac{n}{2}(n+1)$.]

$P(n)$ : $1 + 2 + \cdots + n \leq n^2$.

Base Case : $P(1)$ is $1 \leq 1^2$. True.

50 %

Inductive Step: Assume $P(n)$.

Need to prove $P(n+1)$.

80 %

$$\underbrace{1 + 2 + \cdots + n}_{\text{Use } P(n)} + (n+1) \leq n^2 + n + 1$$

$$\leq n^2 + 2n + 1$$

$$= (n+1)^2.$$

100 %

$$\boxed{P(n+1) : 1 + 2 + \cdots + n + n + 1 \leq (n+1)^2}$$

**Problem 4. Recall the rooted binary tree (RBT) recursive definition:**

① The empty tree $\varepsilon$ is an RBT. **[base case]**

② If $T_1$ and $T_2$ are RBTs with roots $r_1$ and $r_2$, then linking $r_1$ and $r_2$ to a new root $r$ gives a new RBT with root $r$. **[constructor]**

③ No other tree is an RBT. **[minimality]**

**Prove using structural induction that the degree of the root in any rooted binary tree is at most 2.**

Base Case: ~~The root of~~ The empty tree has no root, so trivially true.

50 %

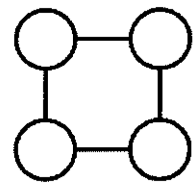
Inductive Step: Consider $T_1$ and $T_2$, which are RBTs. We know $r_1$ and $r_2$ have degree at most 2. The new root has at most 2 children, namely $r_1$ and $r_2$. Done.

100%

Note: the new root may have degree 0 or 1 if either of $T_1$ or $T_2$ is empty.

**Problem 5. The diameter of a graph is the distance between the two vertices that are furthest apart. Recall that $C_n$ denotes a cycle graph with $n$ vertices ($C_4$ is shown below). What is the diameter of $C_n$?**

[You don't need to prove your answer but you need to provide sufficient reasoning to justify the formula.]

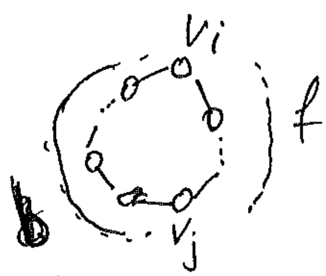


Consider any nodes $v_i$ and $v_j$. Suppose nodes are labeled such that $v_1 \to v_2 \to \cdots \to v_n$. Suppose $j > i$.

50 %

The forward path from $i$ to $j$ goes like $v_i \to v_{i+1} \to \cdots \to v_j$.
The backward path goes $v_j \to v_{j+1} \to \cdots \to v_1 \to \cdots v_i$.
Let $f$ be the length of the forward path and let $b$ be the length of the backward path.
Notice that $f + b = n$.

80%




Tinker:
$f = 1,\ b = n-1 \to dist(i,j) = 1$
$\wedge$
$f = 2,\ b = n-2 \to dist(i,j) = 2$
$\wedge$
$\vdots$

inc.
~~dec.~~

assume $n$ is odd

$f = \frac{n-1}{2},\ b = \frac{n+1}{2} \to dist(i,j) = \frac{n-1}{2}$
$\shortparallel$
$\boxed{f = \frac{n+1}{2},\ b = \frac{n-1}{2} \to dist(i,j) = \frac{n-1}{2}}$
$\wedge$
$f = \frac{n+3}{2},\ b = \frac{n-3}{2} \to dist(i,j) = \frac{n-3}{2}$

dec.

100 %

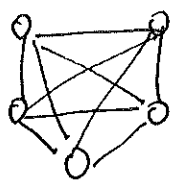
when $n$ is even, max is at $\frac{n}{2}$. $\boxed{\text{Concise answer: } \left\lfloor \frac{n}{2} \right\rfloor}$

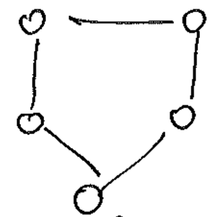
**Problem 6.** A graph is $r$-regular if every vertex has the same degree $r$. Suppose $r$ is even. Show that for all even $r$ and for all $n$ such that $n > r$, there exists an $r$-regular graph with $n$ vertices. Tinker, tinker, tinker.

Tinker.

$r = 4$ (clique) $\qquad$ $r = 2$ (immediate neighbors, i.e. cycle)

$n = 5$:

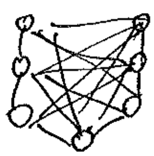


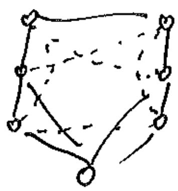


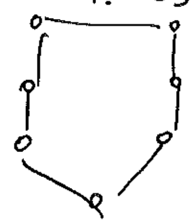
50 %

$r = 6$ (clique) $\qquad$ $r = 4$ (remove 2 "distant" nodes) $\qquad$ $r = 2$ (immediate neighbors, i.e. cycle)

$n = 7$:







[To define distant neighbors,* suppose nodes are labeled $v_1, \dots, v_n$ and measure distance between indices, similar to Problem 5.]

Pattern (for odd $n$):

when $r = n-1$: clique

when $r = n-3$: ~~clique~~ start w/ clique, and remove two "most distant neighbors" for each node

⋮

start w/ previous graph and remove two "most distant neighbors" for each node

80 %

Pattern (for even $n$):

when $r = n-2$: start w/ clique, remove "most distant neighbor" for each node

⋮

same as odd $n$ case

100 %

# Scratch