

# Expected Value of a Sum

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# Reading

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- Malik Magdon-Ismael. Discrete Mathematics and Computing.
  - Chapter 20

# Overview

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- Expected value of a sum
  - Sum of dice
  - Binomial
  - Waiting time
  - Coupon collecting
- Iterated expectation
- Build-up expectation
- Expected value of a product
- Sum of indicators

# Expected Value of a Sum

- Suppose you play the lottery. If you get two tickets, you expect to win two times as much money
- If you flip two coins, you expect to get two times as many H's (1 vs. 0.5)
- **The expected value of a sum is a sum of the expected values.**
- *Theorem [Linearity of Expectation].* Let  $X_1, X_2, \dots, X_k$  be random variables and let  $Z = a_1X_1 + a_2X_2 + \dots + a_kX_k$  be a linear combination of the  $X_i$ 's. Then,

$$\mathbb{E}[Z] = a_1\mathbb{E}[X_1] + a_2\mathbb{E}[X_2] + \dots + a_k\mathbb{E}[X_k]$$

- *Proof.*

$$\begin{aligned}\mathbb{E}[Z] &= \sum_{\omega \in \Omega} (a_1X_1(\omega) + a_2X_2(\omega) + \dots + a_kX_k(\omega)) \cdot \mathbb{P}[\omega] \\ &= a_1 \sum_{\omega \in \Omega} X_1(\omega) \cdot \mathbb{P}[\omega] + a_2 \sum_{\omega \in \Omega} X_2(\omega) \cdot \mathbb{P}[\omega] + \dots + a_k \sum_{\omega \in \Omega} X_k(\omega) \cdot \mathbb{P}[\omega] \\ &= a_1\mathbb{E}[X_1] + a_2\mathbb{E}[X_2] + \dots + a_k\mathbb{E}[X_k]\end{aligned}$$

- QED.

## Expected Value of a Sum, cont'd

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- Summation can be taken inside or pulled outside an expectation
- Constants can be taken inside or pulled outside an expectation

$$\mathbb{E} \left[ \sum_{i=1}^k a_i X_i \right] = \sum_{i=1}^k a_i \mathbb{E}[X_i]$$

# Sum of Dice

- Let  $X$  be the sum of 4 fair dice. What is  $\mathbb{E}[X]$ ?
- Let's list the outcome tree!

$sum$	4	5	6	7	...	24
$\mathbb{P}[sum]$	$\frac{1}{1296}$	$\frac{4}{1296}$	$\frac{10}{1296}$	?	...	$\frac{1}{1296}$

- So the expected value is,

$$\mathbb{E}[X] = 4 \times \frac{1}{1296} + 5 \times \frac{4}{1296} + \dots$$

– Um, this is going to take a while...

- MUCH faster to observe that  $X$  is a sum:

$$X = X_1 + X_2 + X_3 + X_4$$

– where  $X_i$  is the value rolled by die  $i$

– what is the expected value of each  $X_i$ ?

$$\mathbb{E}[X_i] = 3.5$$

- By Linearity of Expectation:

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] + \mathbb{E}[X_4] = 14$$

in general  $n \times 3.5$

# Sum of Dice Exercise

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- **Exercise.** Compute the full PDF for the sum of 4 dice and expected value from the PDF.

# Expected Number of Successes in $n$ Coin Tosses

- Suppose we have  $n$  trials, and the probability of success is  $p$
- Let  $X$  denote the number of successes

$$X = X_1 + X_2 + \cdots + X_n$$

- Each  $X_i$  is a Bernoulli and

$$\mathbb{E}[X_i] = p$$

– Why?

$$\mathbb{E}[X_i] = 1 \times p + 0 \times (1 - p)$$

- By linearity of expectation,

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{E}[X_1 + X_2 + \cdots + X_n] \\ &= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n] \\ &= n \times p\end{aligned}$$

# Expected Waiting Time to $n$ Successes

- Let  $X$  be the waiting time for  $n$  successes with success probability  $p$
- How do we define  $X$  formally?

$$X = X_1 + X_2 + \cdots + X_n$$

- where  $X_1$  is the waiting time until the first success
- where  $X_2$  is the waiting time from the first to the second success
- Each  $X_i$  is a waiting time to *one* success
  - What is  $\mathbb{E}[X_i]$ ?

$$\mathbb{E}[X_i] = \frac{1}{p}$$

- By linearity of expectation:

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{E}[X_1 + X_2 + \cdots + X_n] \\ &= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n] \\ &= \frac{n}{p}\end{aligned}$$

- **Example.** If you are waiting for 3 heads, you have to wait 3-times as long as for 1 head.
- **Exercise.** Compute the expected *square* of the waiting time.

# Coupon Collection

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- When I was a kid, when the Soccer World Cup started, we would try to collect stickers of all the players
  - Each pack had 5 stickers, but there were looooots of repeats
- Kids also would collect country flags from gum packs
  - There were 169 flags in total, but lots of repeats again
  - How many gum-purchases would one have to make on average?
- Let  $X$  be the waiting time to collect all 169 flags
$$X = X_1 + X_2 + \cdots + X_n$$
  - where  $X_1$  is the waiting time until the first success
  - where  $X_2$  is the waiting time from the first to the second success, etc.
- Are probabilities of success the same?
  - Chance of repeats increases after each new flag

# Coupon Collection, cont'd

- Let  $X$  be the waiting time to collect all 169 flags

$$X = X_1 + X_2 + \cdots + X_n$$

- where  $X_1$  is the waiting time until the first success
  - where  $X_2$  is the waiting time from the first to the second success, etc.
- What is  $\mathbb{E}[X_1]$ ?

$$\mathbb{E}[X_1] = 1 = \frac{n}{n}$$

- No repeats
- The probability of the 2<sup>nd</sup> success is  $\frac{n-1}{n}$ , so  $\mathbb{E}[X_2] = \frac{n}{n-1}$
  - The probability of the 3<sup>rd</sup> success is  $\frac{n-2}{n}$ , so  $\mathbb{E}[X_3] = \frac{n}{n-2}$
  - By linearity of expectation:

$$\mathbb{E}[X] = n \times \left( \frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{1} \right) = nH_n \approx n(\ln n + 0.577)$$

- When  $n = 169$ , you expect to buy 965 gum packs! Better have strong teeth!

# Coupon Collection Example

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- **Example.** Cereal box contains 1-of-5 cartoon characters. Collect all to get \$2 rebate
  - Expect to buy about 12 cereal boxes
  - If a cereal box costs \$5, that's a whopping 3.3% discount

# Iterated Expectation

- Here's a convoluted experiment (probabilists like twisted expectations...)
- **Experiment.** Roll a die and let  $X_1$  be the value. Now, roll a second die  $X_1$  times and let  $X_2$  be the sum of these  $X_1$  rolls of the second die.
  - An example outcome is (4; 2,1,2,6)
    - with  $X_1 = 4, X_2 = 11$

- What is  $\mathbb{E}[X_2]$ ?

– Can use the law of total expectation to calculate it. How?

$$\mathbb{E}[X_2] = \mathbb{E}[X_2|X_1 = 1] \times \mathbb{P}[X_1 = 1] + \dots + \mathbb{E}[X_2|X_1 = 6] \times \mathbb{P}[X_1 = 6]$$

- Let's look at another approach. What is the conditional expectation  $\mathbb{E}[X_2|X_1]$ ?

$$\mathbb{E}[X_2|X_1] = X_1 \times 3.5$$

– The RHS is a *function* of  $X_1$ , a random variable. Compute its expectation.

- Another version of the law of total expectation lets us write

$$\begin{aligned}\mathbb{E}[X_2] &= \mathbb{E}_{X_1}[\mathbb{E}[X_2|X_1]] \\ &= \mathbb{E}[X_1] \times 3.5 \\ &= 3.5 \times 3.5 = 12.25\end{aligned}$$

## Build-Up Expectation: Waiting for 2 Hs and 6 Ts

- Probabilists also like waiting!
  - And while they wait, they like to calculate how long they are expected to wait
- Suppose we have a potentially biased coin,  $\mathbb{P}[H] = p$

- Let's introduce the relevant notation:

$$W(k, l) = \mathbb{E}[\text{waiting time to } k \text{ Hs and } l \text{ Ts}]$$

- The first toss is either a H or a T, so by total expectation:

$$\begin{aligned} W(k, l) &= \mathbb{E}[\text{waiting time}|H] \times \mathbb{P}[H] + \mathbb{E}[\text{waiting time}|T] \times \mathbb{P}[T] \\ &= (1 + W(k - 1, l)) \times \mathbb{P}[H] + (1 + W(k, l - 1)) \times \mathbb{P}[T] \end{aligned}$$

$$= 1 + pW(k - 1, l) + (1 - p)W(k, l - 1)$$

- Aha, this is a recursion similar to the candies counting problem!
  - What are the base cases:

$$W(0, l) = \frac{l}{1 - p}$$

$$W(k, 0) = \frac{k}{p}$$

# Build-Up Expectation: Waiting for 2 Hs and 6 Ts, cont'd

- Suppose we have a potentially biased coin,  $\mathbb{P}[H] = p$

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- What are the base cases:

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$$W(k, 0) = \frac{k}{p}$$

		$l$								
		0	1	2	3	4	5	6	7	...
$k$	$W(k, l)$	0	2	4	6	8	10	12	14	...
	0	0	2	4	6	8	10	12	14	...
	1	2								
	2	4								
	...	...								

# Build-Up Expectation: Waiting for 2 Hs and 6 Ts, cont'd

- Suppose we have a potentially biased coin,  $\mathbb{P}[H] = p$

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$$W(k, l) = \mathbb{E}[\text{waiting time to } k \text{ Hs and } l \text{ Ts}]$$

- The first toss is either a H or a T, so by total expectation:

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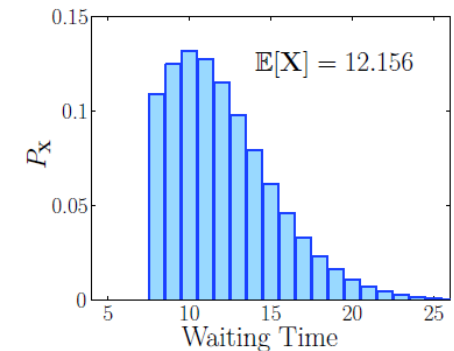
- What are the base cases:

$$W(0, l) = \frac{l}{1 - p}$$

$$W(k, 0) = \frac{k}{p}$$

$l$

$W(k, l)$	0	1	2	3	4	5	6	7	...
0	0	2	4	6	8	10	12	14	...
1	2	3	4.5	6.25	8.13	10.1	12.0	14.0	...
2	4	4.5	5.5	6.88	8.5	10.3	12.2	14.1	...
...	...	...	...	...	...	...	...	...	...



# Expected Value of a Product

- Suppose I roll a single die (pew, simple!)
  - Wait for it...
  - What is the expected value of the squared die roll?

- What are the outcomes?

$$X = 1 \left( \mathbb{P}[X = 1] = \frac{1}{6} \right), \dots, X = 36 \left( \mathbb{P}[X = 36] = \frac{1}{6} \right)$$

$$\mathbb{E}[X^2] = \frac{1}{6} \times 1 + \frac{1}{6} \times 4 + \dots + \frac{1}{6} \times 36 = 15.16$$

- BEWARE!

- Expectation is linear, but it's not quadratic (i.e., it does not distribute inside)

$$\mathbb{E}[X] \times \mathbb{E}[X] = 12.25 \neq \mathbb{E}[X^2] = 15.16$$

- Expectation does distribute inside if the variables are independent!
- Let  $X_1$  and  $X_2$  be two independent rolls

$$\begin{aligned} \mathbb{E}[X_1 X_2] &= \frac{1}{36} (1 + 2 + \dots + 6 + 2 + 4 + \dots + 12 + \dots + 6 + 12 + \dots + 36) \\ &= \frac{441}{36} = 12.25 \\ &= \mathbb{E}[X_1] \mathbb{E}[X_2] = (3.5)^2 \end{aligned}$$

Die 2 Value	6	12	18	24	30	36
	5	10	15	20	25	30
	4	8	12	16	20	24
	3	6	9	12	15	18
	2	4	6	8	10	12
	1	2	3	4	5	6
Die 1 Value						





# Expected Value of a Product, cont'd

- Expected value of a product  $XY$ 
  - In general, the expected product is not a product of expectations.
  - For independent random variables, it is:  $\mathbb{E}[XY] = \mathbb{E}[X] \times \mathbb{E}[Y]$
  - Why?

$$\begin{aligned}\mathbb{E}[XY] &= \sum_{x \in X(\Omega)} \sum_{y \in Y(\Omega)} xy \mathbb{P}[X = x, Y = y] \\ &= \sum_{x \in X(\Omega)} \sum_{y \in Y(\Omega)} xy \mathbb{P}[X = x] \mathbb{P}[Y = y] \\ &= \sum_{x \in X(\Omega)} x \mathbb{P}[X = x] \sum_{y \in Y(\Omega)} y \mathbb{P}[Y = y] \\ &= \mathbb{E}[X] \mathbb{E}[Y]\end{aligned}$$

# Sum of Indicators: Successes in a Random Assignment

- Another fun experiment!
  - Consider 4 people with hats. I take their hats and throw them randomly. What is the expected number of hats that land on a correct head?
    - (Assuming hats only land on heads!)
- Let  $X$  be the number of correct hats when 4 hats randomly land on 4 heads

Hat #				
Person #	①	②	③	④
	$X_1 = 0$	$X_2 = 1$	$X_3 = 1$	$X_4 = 0$

$$X = X_1 + X_2 + X_3 + X_4$$

- What is the distribution of each  $X_i$ ?
  - Bernoulli, with  $\mathbb{P}[X_i = 1] = \frac{1}{4}$ 
    - Hats are randomly distributed
- By linearity of expectation:

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] + \mathbb{E}[X_4] = 1$$

# Sums of Indicators Exercises

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- **Exercise.** What about if there are  $n$  people?
- **Interesting Example (see text).** Apply sum of indicators to breaking of records
- **Instructive Exercise.** Compute the PDF of  $X$  and the expectation from the PDF