

# Infinity

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# Reading

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- Malik Magdon-Ismael. Discrete Mathematics and Computing.
  - Chapter 22

# Summary of Our Stroll Through Discrete Math

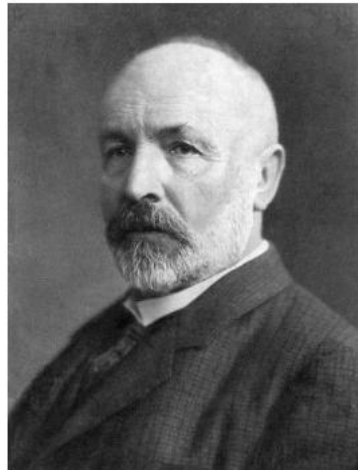
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- Precise statements, proofs and logic.
- **INDUCTION.**
- Recursively defined structures and Induction. (Data structures; PL)
- Sums and asymptotics. (Algorithm analysis)
- Number theory. (Cryptography; probability; fun)
- Graphs. (Relationships/conflicts; resource allocation; routing; scheduling, . . . )
- Counting. (Enumeration and brute force algorithms)
- Probability. (Real world algorithms involve randomness/uncertainty)
  - Inputs arrive in a random order;
  - Randomized algorithms (primality testing, machine learning, routing, conflict resolution . . . )
  - Expected value is a summary of what happens. Variance tells you how good the summary is.

# Today: Infinity

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- Comparing “sizes” of sets: countable.
  - Rationals are countable.
- Uncountable
  - Infinite binary strings.
- What does Infinity have to do with computing?

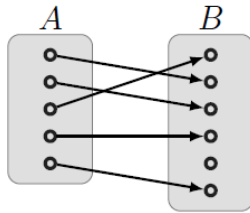


Georg Cantor

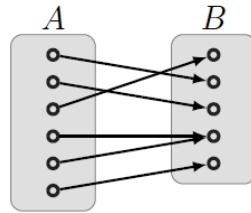


# “Size” of a Set: Cardinality

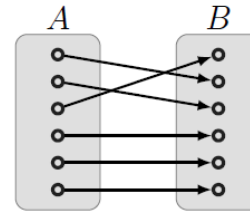
- There’s a reason why small kids use fingers to count
  - They map their intuitive knowledge of 2-fingers to 2 of another object
- You have an *equal* number of fingers on each hand
  - You can map your left hand’s fingers to your right hand’s fingers
- Recall some types of such maps



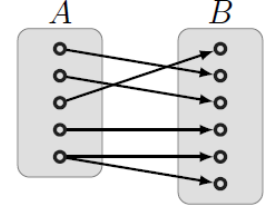
1-1, but **not** onto.  
(injection,  $A \xrightarrow{INJ} B$ )  
 $|A| \leq |B|$



onto; **not** 1-1  
(surjection,  $A \xrightarrow{SUR} B$ )  
 $|A| \geq |B|$

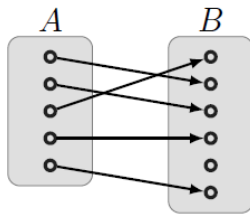


onto **and** 1-1  
(bijection,  $A \xrightarrow{BIJ} B$ )  
 $|A| = |B|$

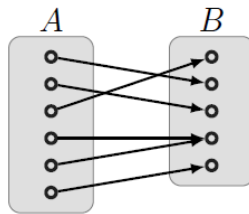


not a function

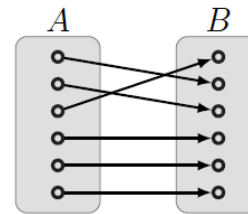
# “Size” of a Set: Cardinality



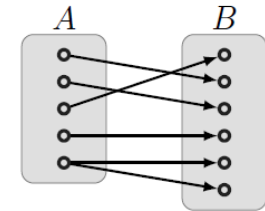
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not a function

- *Cardinality*  $|A|$  (“size”), read “cardinality of  $A$ ,” is the number of elements for finite sets
- In general, we can define the following relations between sets:

$|A| \leq |B|$  iff there is an injection (1-to-1) from  $A$  to  $B$ , i.e.,  $f: A \xrightarrow{INJ} B$

$|A| > |B|$  iff there is no injection (1-to-1) from  $A$  to  $B$

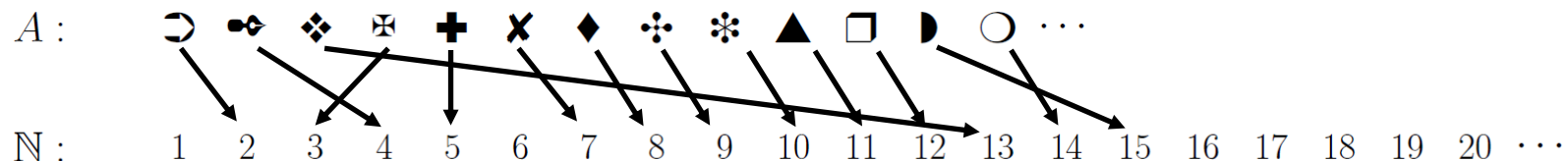
$|A| \geq |B|$  iff there is a surjection (onto) from  $A$  to  $B$ , i.e.,  $f: A \xrightarrow{SUR} B$

$|A| = |B|$  iff there is a bijection (1-1 and onto) from  $A$  to  $B$ , i.e.,  $f: A \xrightarrow{BIJ} B$

$|A| \leq |B|$  and  $|B| \leq |A| \rightarrow |A| = |B|$  [Cantor-Bernstein Theorem]

# A Countable Set's Cardinality is at most $|\mathbb{N}|$

- Suppose we have a finite set  $A = \{a_1, \dots, a_n\}$ 
  - Cardinality is  $|A| = n$  if and only if there is a bijection from  $A$  to  $\{1, \dots, n\}$
  - Can you come up with such a function?
$$f(a_i) = i$$
- For infinite sets: the set  $A$  is countable if  $|A| \leq |\mathbb{N}|$ .
  - Intuitively,  $A$  is “smaller than”  $\mathbb{N}$
  - Sometimes we say  $A$  is at most countable to include both finite and infinite sets that are “smaller than”  $\mathbb{N}$
- To show that  $A$  is countable you must find a 1-to-1 mapping from  $A$  to  $\mathbb{N}$



- You cannot skip over any elements of  $A$ , but you might not use every element of  $\mathbb{N}$

## A Countable Set's Cardinality is at most $|\mathbb{N}|$ , cont'd

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- To prove that a function  $f: A \mapsto \mathbb{N}$  is an injection:
  - Assume  $f$  is *not* an injection. (Proof by contradiction.)
  - This means there is a pair  $x, y \in A$  for which  $x \neq y$  and  $f(x) = f(y)$
  - Use  $f(x) = f(y)$  to prove that  $x = y$ , a contradiction. Hence,  $f$  is an injection

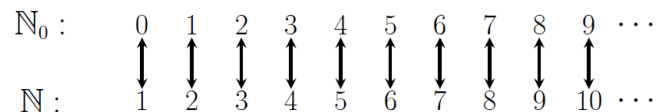
# All Finite Sets are Countable

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- Suppose  $A = \{3,6,8\}$ 
  - To show  $|A| \leq |\mathbb{N}|$ , we give an injection from  $A$  to  $\mathbb{N}$   
 $3 \mapsto 1, 6 \mapsto 23134, 8 \mapsto 8$
- For an arbitrary set  $A = \{a_1, \dots, a_n\}$   
 $a_1 \mapsto 1, a_2 \mapsto 2, \dots, a_n \mapsto n$

# Non-negative integers $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ are countable

- How can this be??
  - I know for a fact that  $\mathbb{N}_0$  contains every element in  $\mathbb{N}$ , plus an extra 0
  - It's clearly bigger!!
  - But “bigger” means something else for infinite sets
- To prove that  $|\mathbb{N}_0| \leq |\mathbb{N}|$ , we need an injection  $f: \mathbb{N}_0 \xrightarrow{INJ} \mathbb{N}$ 
  - Ideas?
  - Let's try  $f(x) = x + 1, \forall x \in \mathbb{N}_0$
- *Proof.*
  - Assume  $f$  is not an injection. So, there are  $x \neq y$  in  $\mathbb{N}_0$  with  $f(x) = f(y)$ :  
$$x + 1 = f(x) = f(y) = y + 1$$
  - But that means  $x + 1 = y + 1$ , i.e.,  $x = y$ . Contradiction.
- Also, we know that  $|\mathbb{N}| \leq |\mathbb{N}_0|$  since  $\mathbb{N} \subseteq \mathbb{N}_0$
- By the Cantor-Bernstein Theorem,  $\mathbb{N} = \mathbb{N}_0$



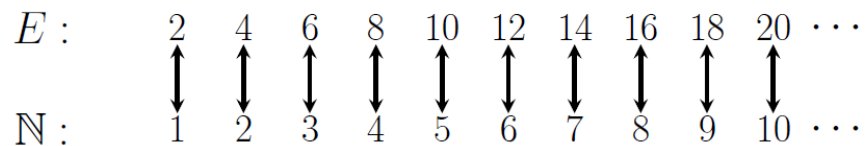
# Positive Even Numbers and Integers and Countable

- Huh? The even numbers are exactly half of all natural numbers!!

$$E = \{2, 4, 6, \dots\}, \text{ so surely } |E| = \frac{1}{2} |\mathbb{N}|!!$$

- Turns out, not quite. Can you see a bijection?

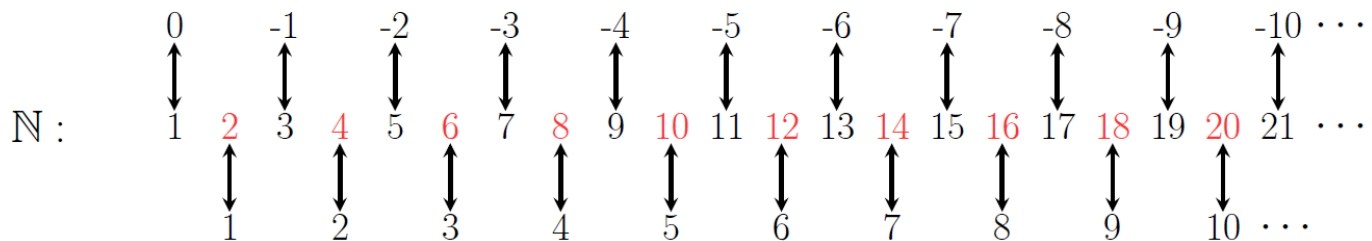
- The bijection  $f(x) = \frac{1}{2}x$  proves  $|E| = |\mathbb{N}|$



- OK, fine, but all integers?? It has to be the case that  $|\mathbb{Z}| = 2|\mathbb{N}|!!$

- Is there a bijection? ☺

- Recall the integers are  $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$



- **Exercise.** What is a mathematical formula for the bijection?

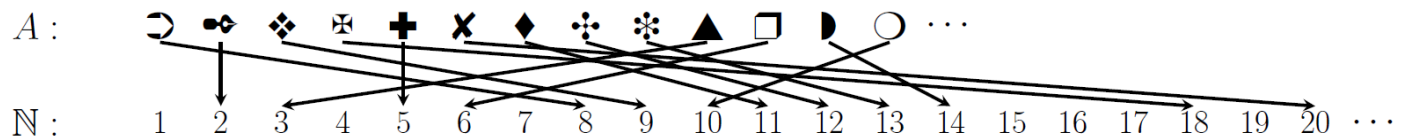
# Every Countable Set Can Be “Listed”

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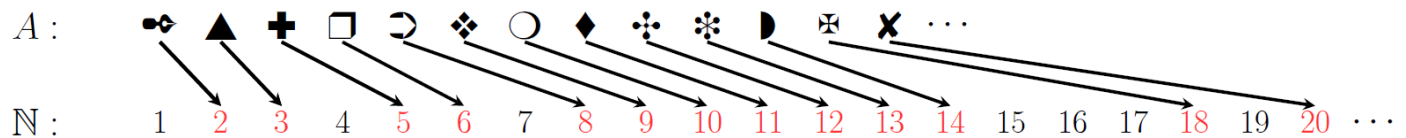
- What does it mean for a set to be “listed”?
  - You know the exact position of each element in the set
  - Regardless of whether the set is finite or infinite
- For example:
  - $\{3,6,8\}$  is a list (why?)
    - (because it’s a finite set)
  - $E = \{2,4,6, \dots\}$  is a list (why?)
    - element  $i$  is just  $2i$
  - What about  $\mathbb{Z}$ ?
    - Suppose I represent it as  $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ 
      - Unclear what the indices are
    - Suppose I represent it as  $\{0, -1, 1, -2, 2, \dots\}$ 
      - Element 1 is 0, o.w. it is  $-i/2$  (if  $i$  is even),  $(i - 1)/2$  (if  $i$  is odd)

# Every Countable Set Can Be “Listed”, cont’d

- Suppose I give you the following mapping between sets  $A$  and  $\mathbb{N}$



- How do I “list”  $A$ ?
- Order elements according to their assigned value



- In general, a set can be “listed” if
  - Different elements are assigned to different list-positions.
  - We can determine the list-position of *any* element in the set.

# Union of Two Countable Sets is Countable

- Consider two countable sets,  $A$  and  $B$ 
  - They are countable, so I can write  $A = \{a_1, a_2, a_3, \dots\}$  and  $B = \{b_1, b_2, b_3, \dots\}$

- Now, let's look at the union

$$A \cup B = \{a_1, a_2, a_3, \dots, b_1, b_2, b_3, \dots\}$$

- Hm, how do I show this is countable?
  - Can't use “...” twice
  - How do I reorder terms?
  - Need to know the position of each  $b_i$
- Here's a better reordering:
$$A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, \dots\}$$
  - Now I know the position of each element
  - List-position of  $a_i$  is  $2i - 1$
  - List-position of  $b_i$  is  $2i$
- **Exercise.** Get a list of  $\mathbb{Z}$  with  $A = \{0, -1, -2, -3, \dots\}$  and  $B = \{1, 2, 3, \dots\}$  using union.

## Rationals are Countable: $|\mathbb{Q}| = |\mathbb{N}|$

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- OK, this one is very surprising!
  - There are **infinitely** many rationals between every two integers!
  - The rationals are dense (there is a rational between any two rationals)!
  - Natural numbers are not!
  - Also, the set of rationals can be expressed as the product of integers and natural numbers:
    - $|\mathbb{Q}| = |\mathbb{N}| \times |\mathbb{Z}|$ , so of course  $|\mathbb{Q}| \gg |\mathbb{N}|??$
- Well, let's see...

# Rationals are Countable: $|\mathbb{Q}| = |\mathbb{N}|$ , cont'd

		$\mathbb{Z}$								
		0	1	-1	2	-2	3	-3	4	-4
$\mathbb{N}$	1	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{-1}{1}$	$\frac{2}{1}$	$\frac{-2}{1}$	$\frac{3}{1}$	$\frac{-3}{1}$	$\frac{4}{1}$	$\frac{-4}{1}$
	2	$\frac{0}{2}$	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{2}{2}$	$\frac{-2}{2}$	$\frac{3}{2}$	$\frac{-3}{2}$	$\frac{4}{2}$	$\frac{-4}{2}$
	3	$\frac{0}{3}$	$\frac{1}{3}$	$\frac{-1}{3}$	$\frac{2}{3}$	$\frac{-2}{3}$	$\frac{3}{3}$	$\frac{-3}{3}$	$\frac{4}{3}$	$\frac{-4}{3}$
	4	$\frac{0}{4}$	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{2}{4}$	$\frac{-2}{4}$	$\frac{3}{4}$	$\frac{-3}{4}$	$\frac{4}{4}$	$\frac{-4}{4}$
	5	$\frac{0}{5}$	$\frac{1}{5}$	$\frac{-1}{5}$	$\frac{2}{5}$	$\frac{-2}{5}$	$\frac{3}{5}$	$\frac{-3}{5}$	$\frac{4}{5}$	$\frac{-4}{5}$
	...	...	...	...	...	...	...	...	...	...

- How do I “list” all rationals?
  - I need a list that visits each rational *exactly* once!
  - I need to know each rational’s position *exactly*

$$\text{Rational Values} = \left\{ \frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{0}{2}, \frac{0}{3}, \frac{1}{3}, \dots \right\}$$

$$|\mathbb{Q}| \leq |\{\text{Rational Values}\}| \leq |\mathbb{N}|$$

- **Exercise.** What is a mathematical formula for the list-position of  $z/n \in \mathbb{Q}$ ?

# Programs are Countable

- Programs are finite binary strings. We show that all finite binary strings  $\mathcal{B}$  are countable

– How do I list them?

- Start with the empty string (duh...)
- Then list all strings of length 1, length 2, etc.

$$\mathcal{B} = \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, \dots\}$$

- I now know the exact position of every string!
- **Exercise.** What is the list-position of 0110?
- **Exercise.** For the  $(k + 1)$ -bit string  $b = b_k b_{k-1} \dots b_1 b_0$ , define the string's numerical value:

$$value(b) = b_0 \cdot 2^0 + b_1 \cdot 2^1 + \dots + b_{k-1} 2^{k-1} + b_k 2^k$$

– Show:

$$\text{list-position of } b = 2^{\text{length}(b)} + value(b)$$

- Wait a second... We keep seeing larger and larger sets that are countable!!

$$\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Q}, \mathcal{B}$$

– SURELY EVERYTHING IS COUNTABLE!?!

# Infinite Binary Strings Are Uncountable!

- One of the most cool results in computational theory
- Cantor's Diagonal Argument: Assume there is a list of *all* infinite binary strings

$b_1$ : 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 ...  
 $b_2$ : 0 0 1 1 0 1 0 0 1 0 0 0 0 1 0 ...  
 $b_3$ : 1 1 0 0 0 0 1 0 0 0 0 1 1 0 0 ...  
 $b_4$ : 1 0 1 0 0 1 0 0 0 0 1 0 0 0 0 ...  
 $b_5$ : 0 1 1 0 1 0 1 0 0 0 0 0 0 0 0 ...  
 $b_6$ : 0 1 0 1 1 0 0 0 0 1 0 0 0 0 0 ...  
 $b_7$ : 0 0 1 0 0 1 0 0 0 0 0 0 0 0 1 ...  
 $b_8$ : 0 0 1 0 1 1 0 1 0 0 0 0 1 0 0 ...  
 $b_9$ : 0 0 0 0 1 1 0 0 0 0 1 0 0 0 0 ...  
 $b_{10}$ : 1 0 1 1 1 0 1 0 0 1 1 0 0 0 0 ...  
⋮

- We'll now show that there exists a string that cannot be in that list!
- Look at the (red) diagonal string:

$$b = 0000100101 \dots$$

- What's so special about this string?

– Let's flip the bits

$$\bar{b} = 1111011010 \dots$$

– This string is not in the list!

– Differs from each string  $b_i$  in position  $i$

# The Real Numbers are Uncountable

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- Every real between 0 and 1 has an infinite binary representation and every infinite binary string evaluates to a real number

$$\text{e.g., } 0.001111111111111111 \dots = \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \dots = \frac{1}{4}$$

- This means

$$|\{\text{reals in } [0,1]\}| = |\{\text{infinite binary strings}\}| > |\mathbb{N}|$$

- Aha, found one!
- **Brain-breaking exercise (Continuum Hypothesis).** Prove that there is no set  $\mathcal{R}$  s.t.

$$|\{\text{reals in } [0,1]\}| > |\mathcal{R}| > |\mathbb{N}|$$

# Infinity and Computing

- Cantor took on the abstract beast Infinity. (1874)
- ~60 years later, Alan Turing asked the abstract question: What can we compute? (1936)

- For example, consider the set of binary functions  $f$  defined on  $\mathbb{N}$

$n:$	1	2	3	4	5	6	7	8	9	10	...
$f(n):$	0	1	1	0	1	0	0	0	1	1	...

- Turns out the set of all such functions is uncountable (why?)
  - Corresponds to the set of all infinite binary strings
- Every program is a finite binary string. For example:

```
int main(); //a program that does nothing
```

- This program corresponds to the finite binary string (ASCII code)

```
0110100101101110011101000010000001101101011000010110100101101110001010000010100100111011
```

- So, the number of programs is countable
  - But the number of functions is uncountable!
- There are many more functions than we can write/compute!

# Infinity and Computing, cont'd

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- There are MANY MANY functions that cannot be computed by programs!
- Are there interesting, useful functions that cannot be computed by programs?