Quiz 1

60 Minutes

First Name: ____________________

Last Name: ____________________

RIN: ____________________

Section: ____________________

NO COLLABORATION or electronic devices.
Any violations will result in an F.
No questions allowed during the test unless you think there is a mistake.

GOOD LUCK!

Circle at most one answer per question.
10 points for each correct answer.

You **MUST** show **CORRECT** work to get credit.
Correct answers with no explanation will get a 0.

Final Score: ______ / 200

1. What is a simpler expression for the set $A = \{n \mid n = k + m, k \in \mathbb{Z}, m \in \mathbb{N}\}$?

   A $A = \{n \mid n \in \mathbb{N}\}$

   B $A = \{n \mid n \in \mathbb{Z}\}$

   C $A = \{n \mid n = 1 + k, k \in \mathbb{N}\}$

   D $A = \{n \mid n = -1 + k, k \in \mathbb{N}\}$

   E None of the above.

2. Define sets $A = \{n \mid n = 2k, k \in \mathbb{Z}\}$ and $B = \{n \mid n \in \mathbb{N}\}$. Which of the following is true?

   A $A \subseteq B$

   B $B \subseteq A$

   C $A = B$

   D $A \cap B = \{n \mid n = 2k, k \in \mathbb{N}\}$

   E None of the above.

3. What is the value of the expression $(p \wedge q) \rightarrow (\neg p \vee q)$?

   A Depends on the value of p.

   B Depends on the value of q.

   C Always true.

   D Always false.

   E None of the above.

4. IF it snows on a given day, THEN the temperature on that day is below 35. There is snow on the ground and the temperature today is –15. What do we know?

   A It is snowing today.

   B It snowed yesterday.

   C On the day when it snowed, the temperature was below 35.

   D It snows every day.

   E None of the above.

5. What is another way to write the expression $(p \wedge q) \rightarrow \neg p$?

   A $\neg p \wedge q$

   B $\neg p \wedge \neg q$

   C $\neg p \vee q$

   D $\neg p \vee \neg q$

   E None of the above.

6. Suppose I want to prove $p \to q$. Which of the following proof techniques will work?

- A Assume p and show that it leads to q being false.
- B Assume q is true and show that p must be true.
- C Assume q is false and show that p must be true.
- D Use a proof by contradiction to show that q can never be true.
- E None of the above.

7. Which of the following claims is NOT true?

- A $\forall x \in \mathbb{N} : \exists y \in \mathbb{R} : \sqrt{x} = y$
- B $\forall x \in \mathbb{Z} : \exists y \in \mathbb{R} : (x \geq 0 \to \sqrt{x} = y)$
- C $\forall x \in \mathbb{Q} : \exists y \in \mathbb{R} : (x \geq 0 \to \sqrt{x} = y)$
- D $\forall x \in \mathbb{N} : \exists y \in \mathbb{N} : \sqrt{x} = y$
- E They are all true.

8. Consider the set $\mathbb{Q}_+ = \{x \mid x > 0, x \in \mathbb{Q}\}$ of positive rational numbers. Suppose I use the following argument to prove that $\mathbb{Q}_+$ has a minimum element: since each rational number can be written as p/q where $q \in \mathbb{N}$, from the principle of well-ordering it follows that $\mathbb{Q}_+$ has a minimum element. What is wrong with this argument?

- A Not all elements of $\mathbb{Q}_+$ can be written as p/q.
- B The principle of well-ordering does not apply to infinite sets.
- C Although the principle of well-ordering can be used to conclude that there is a minimum denominator q, this does not imply that $\mathbb{Q}_+$ has a minimum.
- D The principle of well-ordering is an axiom and cannot be used in proofs.
- E Nothing is wrong: $\mathbb{Q}_+$ has a minimum element!

9. Suppose I try to prove $n^2 + 10 \geq 2^n, \forall n \geq 1$ using induction. What goes wrong?

- A The base case is false.
- B I need three base cases.
- C I cannot prove $P(n) \to P(n+1), \forall n \geq 1$.
- D I cannot prove $P(n+1) \to P(n), \forall n \geq 1$.
- E Nothing goes wrong because the claim is true.

10. How would you *disprove* the claim: $n^4 > 2^n, \forall n \geq 3$?

- A Show that $n^4 > 2^n$ for some $n < 3$.
- B Show that $n^4 > 2^n$ for all $n \geq 3$.
- C Show that $n^4 \leq 2^n$ for some $n < 3$.
- D Show that $n^4 \leq 2^n$ for some $n \geq 3$.
- E None of the above.

11. Let $S_n = 2 + 4 + \cdots + n$ be the sum of all even numbers up to n. What is S_{100}?

   - A 50×99
   - B 5050
   - C 5100
   - D 2550
   - E None of the above.

12. Consider the set $\mathcal{S}$ defined as follows: (1) Base case: $4 \in \mathcal{S}$; (2) Constructor: $x \in \mathcal{S} \to x^2 \in \mathcal{S}$. Which of the following is another definition of the set $\mathcal{S}$?

   - A $\mathcal{S} = \{s \mid s = 2^n, n \in \mathbb{N}\}$
   - B $\mathcal{S} = \{s \mid s = 2^n, n \in \mathbb{Z}\}$
   - C $\mathcal{S} = \{s \mid s = 2^{2n}, n \in \mathbb{N}\}$
   - D $\mathcal{S} = \{s \mid s = 2^{2^n}, n \in \mathbb{N}\}$
   - E None of the above.

13. Define the predicate $P(n) : n + 5 \leq n^2$. For which n is $P(n)$ true?

   - A $n \geq 1$
   - B $n \geq 2$
   - C $n \geq 3$
   - D Only for $3 \leq n \leq 300$
   - E None of the above.

14. As part of a proof by induction, I showed $P(1)$ and $P(2)$ are true, and I also showed that for all $n \geq 3$, there exist $i, j < n$ such that $(P(i) \wedge P(j)) \to P(n)$. Is this sufficient proof that $\forall n \geq 1 : P(n)$?

   - A No, because I didn't prove $P(n) \to P(n+1)$
   - B No, because i and j may not always exist
   - C No, because I didn't prove the base case
   - D Yes, it follows from strong induction. Define $Q(n) = P(1) \wedge \cdots \wedge P(n)$. I essentially showed $Q(n) \to Q(n+1)$ since $P(i)$ and $P(j)$ are part of the induction hypothesis.
   - E None of the above.

15. What can we say about this claim: $\forall x \in \mathbb{R} : \exists y \in \mathbb{R} : xy = y$?

   - A True
   - B False
   - C Depends on the value of x
   - D Depends on the value of y
   - E None of the above.

16. What is another expression for the set $A \cap (A \cup B)$?

- [A] $A \cap B$
- [B] $A \cup B$
- [C] A
- [D] $A \cap \overline{B}$
- [E] None of the above.

17. What is a formal way to define a square of side 1 centered at the origin of $\mathbb{R}^2$?

- [A] $\mathbb{S} = \{(x, y) \mid x \leq 1, y \leq 1, (x, y) \in \mathbb{R}^2\}$
- [B] $\mathbb{S} = \{(x, y) \mid |x| \leq 1, |y| \leq 1, (x, y) \in \mathbb{R}^2\}$
- [C] $\mathbb{S} = \{(x, y) \mid x \leq 1/2, y \leq 1/2, (x, y) \in \mathbb{R}^2\}$
- [D] $\mathbb{S} = \{(x, y) \mid |x| \leq 1/2, |y| \leq 1/2, (x, y) \in \mathbb{R}^2\}$
- [E] None of the above.

18. Let $A_n = A_{n-1}^2$ and $A_0 = 3$. What is a general formula for A_n for $n \geq 0$?

- [A] $A_n = 2^{3n}$
- [B] $A_n = 3^{2n}$
- [C] $A_n = 2^{2^n}$
- [D] $A_n = 3^{2^n}$
- [E] None of the above.

19. Suppose I create a new type of rooted tree, called rooted tertiary tree (RTT), as follows:

① The tree with one vertex is an RTT. **[base case]**

② If T_1, T_2 and T_3 are RTTs with roots r_1, r_2 and r_3, then linking r_1, r_2 and r_3 to a new root r gives a new RTT with root r. **[constructor]**

③ No other tree is an RTT. **[minimality]**

What do we know about RTTs (recall RBT stands for a rooted binary tree)?

- [A] If a vertex has children, then that vertex has three children.
- [B] All RBTs are also RTTs.
- [C] All RTTs have an odd number of vertices.
- [D] All RTTs have an even number of vertices.
- [E] All of the above.

20. Suppose a rooted tertiary tree has 7 vertices. How many links does the tree have?

[A] 6 [B] 7 [C] 8 [D] 9 [E] It cannot be determined.

Scratch