Quiz 1

60 Minutes

First Name: SOLUTIONS

Last Name: ____________

RIN: ____________

Section: ____________

NO COLLABORATION or electronic devices.
Any violations will result in an F.
No questions allowed during the test unless you think there is a mistake.

GOOD LUCK!

Circle at most one answer per question.
10 points for each correct answer.

You **MUST** show **CORRECT** work to get credit.
Correct answers with no explanation will get a 0.

Final Score: ______ / 200

1. What is a simpler expression for the set $A = \{n \mid n = k + m, k \in \mathbb{Z}, m \in \mathbb{N}\}$?

   $\boxed{\text{A}}$ $A = \{n \mid n \in \mathbb{N}\}$
   $\boxed{\text{B}}$ $A = \{n \mid n \in \mathbb{Z}\}$
   $\boxed{\text{C}}$ $A = \{n \mid n = 1 + k, k \in \mathbb{N}\}$
   $\boxed{\text{D}}$ $A = \{n \mid n = -1 + k, k \in \mathbb{N}\}$
   $\boxed{\text{E}}$ None of the above.

Pick any $z \in \mathbb{Z}$.
We know $z = \underbrace{(z-1)}_{\mathbb{Z}} + \underbrace{1}_{\mathbb{N}}$

2. Define sets $A = \{n \mid n = 2k, k \in \mathbb{Z}\}$ and $B = \{n \mid n \in \mathbb{N}\}$. Which of the following is true?

   $\boxed{\cancel{\text{A}}}$ $A \subseteq B$ $-2 \notin B$
   $\boxed{\cancel{\text{B}}}$ $B \subseteq A$ $1 \notin A$
   $\boxed{\cancel{\text{C}}}$ $A = B$
   $\boxed{\text{D}}$ $A \cap B = \{n \mid n = 2k, k \in \mathbb{N}\}$
   $\boxed{\text{E}}$ None of the above.

3. What is the value of the expression $(p \wedge q) \to (\neg p \vee q)$?

   $\boxed{\text{A}}$ Depends on the value of p.
   $\boxed{\text{B}}$ Depends on the value of q.
   $\boxed{\text{C}}$ Always true.
   $\boxed{\text{D}}$ Always false.
   $\boxed{\text{E}}$ None of the above.

We know
$(p \Rightarrow q) \equiv \neg p \vee q$
$(p \wedge q) \Rightarrow (\neg p \vee q) \equiv$
$\equiv \neg p \vee \underline{\neg q} \vee \neg p \vee \underline{q} \equiv T$

4. IF it snows on a given day, THEN the temperature on that day is below 35. There is snow on the ground and the temperature today is –15. What do we know?

   $\boxed{\cancel{\text{A}}}$ It is snowing today.
   $\boxed{\cancel{\text{B}}}$ It snowed yesterday.
   $\boxed{\text{C}}$ On the day when it snowed, the temperature was below 35.
   $\boxed{\text{D}}$ It snows every day.
   $\boxed{\text{E}}$ None of the above.

5. What is another way to write the expression $(p \wedge q) \to \neg p$?

   $\boxed{\text{A}}$ $\neg p \wedge q$
   $\boxed{\text{B}}$ $\neg p \wedge \neg q$
   $\boxed{\text{C}}$ $\neg p \vee q$
   $\boxed{\text{D}}$ $\neg p \vee \neg q$
   $\boxed{\text{E}}$ None of the above.

$(p \wedge q) \Rightarrow \neg p \equiv \neg p \vee \neg q \vee \neg p$

6. Suppose I want to prove $p \to q$. Which of the following proof techniques will work?

p	q	$p \Rightarrow q$	
F	F	T	
F	T	T	
T	F	F	Avoid
T	T	T	

A Assume p and show that it leads to q being false.
B Assume q is true and show that p must be true.
C Assume q is false and show that p must be true.
D Use a proof by contradiction to show that q can never be true.
E None of the above.

7. Which of the following claims is NOT true?

A $\forall x \in \mathbb{N} : \exists y \in \mathbb{R} : \sqrt{x} = y$
B $\forall x \in \mathbb{Z} : \exists y \in \mathbb{R} : (x \geq 0 \to \sqrt{x} = y)$
C $\forall x \in \mathbb{Q} : \exists y \in \mathbb{R} : (x \geq 0 \to \sqrt{x} = y)$
D $\forall x \in \mathbb{N} : \exists y \in \mathbb{N} : \sqrt{x} = y$ $\quad 2 \in \mathbb{N}, \ \sqrt{2} \notin \mathbb{N}.$
E They are all true.

8. Consider the set $\mathbb{Q}_+ = \{x \mid x > 0, x \in \mathbb{Q}\}$ of positive rational numbers. Suppose I use the following argument to prove that $\mathbb{Q}_+$ has a minimum element: since each rational number can be written as p/q where $q \in \mathbb{N}$, from the principle of well-ordering it follows that $\mathbb{Q}_+$ has a minimum element. What is wrong with this argument?

A Not all elements of $\mathbb{Q}_+$ can be written as p/q.
B The principle of well-ordering does not apply to infinite sets.
C Although the principle of well-ordering can be used to conclude that there is a minimum denominator q, this does not imply that $\mathbb{Q}_+$ has a minimum.
D The principle of well-ordering is an axiom and cannot be used in proofs.
E Nothing is wrong: $\mathbb{Q}_+$ has a minimum element!

9. Suppose I try to prove $n^2 + 10 \geq 2^n, \forall n \geq 1$ using induction. What goes wrong?

A The base case is false.
B I need three base cases.
C I cannot prove $P(n) \to P(n+1), \forall n \geq 1$.
D I cannot prove $P(n+1) \to P(n), \forall n \geq 1$.
E Nothing goes wrong because the claim is true.

Exponentials grow faster than polynomials.

10. How would you *disprove* the claim: $n^4 > 2^n, \forall n \geq 3$?

A Show that $n^4 > 2^n$ for some $n < 3$.
B Show that $n^4 > 2^n$ for all $n \geq 3$.
C Show that $n^4 \leq 2^n$ for some $n < 3$.
D Show that $n^4 \leq 2^n$ for some $n \geq 3$.
E None of the above.

The negation of $\forall n \geq 3 : n^4 > 2^n$ is $\exists n \geq 3 : n^4 \leq 2^n$

$S_{100} = \frac{\cancel{4}}{\cancel{4}} \cdot 100 \cdot 102 = 50 \times 51 = 2550$

SHOW WORK

11. Let $S_n = 2 + 4 + \cdots + n$ be the sum of all even numbers up to n. What is S_{100}?

A 50×99
B 5050
C 5100
D 2550
E None of the above.

Use Gauss's technique.

$S_n = 2 + 4 + \cdots + n-2 + n$

$S_n = n + n-2 + \cdots + \cancel{4} + \cancel{2}\,2$

$2S_n = (n+2) \cdot \frac{n}{2} \rightarrow \boxed{S_n = \frac{1}{4} \cdot n \cdot (n+2)}$

SHOW WORK

12. Consider the set $\mathcal{S}$ defined as follows: (1) Base case: $4 \in \mathcal{S}$; (2) Constructor: $x \in \mathcal{S} \rightarrow x^2 \in \mathcal{S}$.
Which of the following is another definition of the set $\mathcal{S}$?

A $\mathcal{S} = \{s \mid s = 2^n, n \in \mathbb{N}\}$
B $\mathcal{S} = \{s \mid s = 2^n, n \in \mathbb{Z}\}$
C $\mathcal{S} = \{s \mid s = 2^{2n}, n \in \mathbb{N}\}$
D $\mathcal{S} = \{s \mid s = 2^{2^n}, n \in \mathbb{N}\}$
E None of the above.

Base case: $4 = 2^2 = 2^{2^1}$

Tinker: $2^2 \in S \Rightarrow 2^4 \in S \Rightarrow 2^8 \in S$.

Inductive step: $P(n): 2^{2^n} \in S$

$(2^{2^n})^2 = 2^{2 \cdot 2^n} = 2^{2^{n+1}}$ $(P(n+1))$. V.

SHOW WORK

13. Define the predicate $P(n) : n + 5 \leq n^2$. For which n is $P(n)$ true?

~~A~~ $n \geq 1$
~~B~~ $n \geq 2$
C $n \geq 3$
D Only for $3 \leq n \leq 300$
E None of the above.

$6 \not\leq 1$
$7 \not\leq 4$

Base case: $P(3)$: $8 \leq 9$. V.

Inductive step: $P(n) \Rightarrow P(n+1)$.

$n+5+1 \leq n^2 + 1 \leq n^2 + 2n + 1$

for $n \geq 3$.

14. As part of a proof by induction, I showed $P(1)$ and $P(2)$ are true, and I also showed that for all $n \geq 3$, there exist $i, j < n$ such that $(P(i) \wedge P(j)) \rightarrow P(n)$. Is this sufficient proof that $\forall n \geq 1 : P(n)$?

A No, because I didn't prove $P(n) \rightarrow P(n+1)$
B No, because i and j may not always exist
C No, because I didn't prove the base case
D Yes, it follows from strong induction. Define $Q(n) = P(1) \wedge \cdots \wedge P(n)$. I essentially showed $Q(n) \rightarrow Q(n+1)$ since $P(i)$ and $P(j)$ are part of the induction hypothesis.
E None of the above.

15. What can we say about this claim: $\forall x \in \mathbb{R} : \exists y \in \mathbb{R} : xy = y$?

A True
B False
C Depends on the value of x
D Depends on the value of y
E None of the above.

$\forall x \in \mathbb{R}$, pick $y = 0$.

Then $0x = 0$.

16. What is another expression for the set $A \cap (A \cup B)$?

A $A \cap B$

B $A \cup B$

C A

D $A \cap \overline{B}$

E None of the above.

$$A \cap (A \cup B) = \overbrace{A \cap B}^{\text{subset of } A} \cup \underbrace{A \cap A}_{A}$$

Union of A and a subset of A is just A.

17. What is a formal way to define a square of side 1 centered at the origin of $\mathbb{R}^2$?

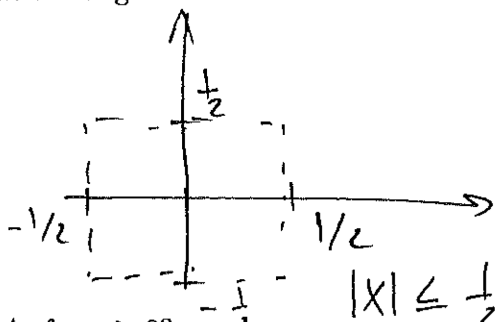
A $\mathbb{S} = \{(x, y) \mid x \leq 1, y \leq 1, (x, y) \in \mathbb{R}^2\}$

B $\mathbb{S} = \{(x, y) \mid |x| \leq 1, |y| \leq 1, (x, y) \in \mathbb{R}^2\}$

C $\mathbb{S} = \{(x, y) \mid x \leq 1/2, y \leq 1/2, (x, y) \in \mathbb{R}^2\}$

D $\mathbb{S} = \{(x, y) \mid |x| \leq 1/2, |y| \leq 1/2, (x, y) \in \mathbb{R}^2\}$

E None of the above.



$|x| \leq \frac{1}{2}$ AN[D]
$|y| \leq \frac{1}{2}$

SHOW WORK

18. Let $A_n = A_{n-1}^2$ and $A_0 = 3$. What is a general formula for A_n for $n \geq 0$?

A $A_n = 2^{3n}$

B $A_n = 3^{2n}$

C $A_n = 2^{2^n}$

D $A_n = 3^{2^n}$

E None of the above.

Base case: $P(0): 3^{2^0} = 3$. ✓.

Inductive step: $P(n): A_n = 3^{2^n}$

$$A_{n+1} = \left(3^{2^n}\right)^2 = 3^{2 \cdot 2^n} = \underbrace{3^{2^{n+1}}}_{P(n+1)}.$$

19. Suppose I create a new type of rooted tree, called rooted tertiary tree (RTT), as follows:

① The tree with one vertex is an RTT. **[base case]**

② If T_1, T_2 and T_3 are RTTs with roots r_1, r_2 and r_3, then linking r_1, r_2 and r_3 to a new root r gives a new RTT with root r. **[constructor]**

③ No other tree is an RTT. **[minimality]**

SHOW WORK

What do we know about RTTs (recall RBT stands for a rooted binary tree)?

A If a vertex has children, then that vertex has three children.

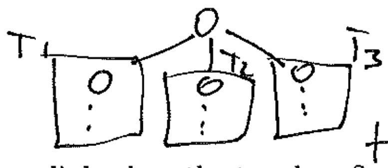
B All RBTs are also RTTs.

C All RTTs have an odd number of vertices.

D All RTTs have an even number of vertices.

E All of the above.

No empty trees. So constructor looks like



. Base case is trivially true since vertex has no children

20. Suppose a rooted tertiary tree has 7 vertices. How many links does the tree have?

A 6 B 7 C 8 D 9 E It cannot be determined.

#links = # vertices − 1. Base case: 1 vertex, 0 links

Inductive step: add 3 links and 1 vertex: $L = (n_1 - 1) + (n_2 - 1) + (n_3 - 1) + 3 = n - 1 - 3 + 3 = n - 1$

Scratch