Quiz 2

60 Minutes

First Name: SOLUTIONS

Last Name: ______

RIN: ______

NO COLLABORATION or electronic devices.
Any violations will result in an F.
No questions allowed during the test unless you think there is a mistake.

GOOD LUCK!

Circle at most one answer per question.
10 points for each correct answer.

For [**Show Work**] problems, you **MUST** show correct work to get credit.
Correct answers with no explanation will get a 0.

Final Score: ______ / 200

1. When we grade FOCS quizzes, we order 3 pizzas. If the pizzeria has 10 types of pizza on the menu, how many different orders can we make (for example, {two pepperoni and one margherita} is the same order as {one margherita and two pepperoni})?

   A $\binom{10}{2}$
   B $10 \times 9 \times 8$
   C $\binom{10}{8}$
   D 10!
   E None of the above.

   Goody bag problem.
   Bags of size 3 : n
   10 types of candy : k $\rightarrow \binom{12}{9}$

2. **[Show Work]** Suppose you are ordering a build-your-own pizza, where you have to choose a crust type, a base type and two different toppings. You have a choice of two types of crust, two types of base and 10 toppings. How many build-your-own pizzas are possible?

   A $\binom{10}{2}$
   B 45
   C 90
   D 180
   E None of the above.

   Product rule:
   $$\underset{\text{crust}}{2} \times \underset{\text{base}}{2} \times \underset{\text{toppings}}{\binom{10}{2}} = 180$$
   $\binom{10}{2} = 45$

3. Suppose FOCS has 100 students. If I order students randomly, how many top 5 orderings are possible (assuming all grades are different)?

   A 100!
   B $\binom{100}{5}$
   C $100 \times 99 \times 98 \times 97 \times 96$
   D 200
   E None of the above.

   Same as # of top-3 finishers in a race of 10.

4. Let X and Y be random variables, and let Ω be the set of all outcomes. When is the following true: $\sum_{x \in X(\Omega)} \mathbb{P}[X = x] = \sum_{y \in Y(\Omega)} \mathbb{P}[Y = y]$?

   A Never.
   B Always.
   C Only when X and Y are the same variable.
   D There is not enough information to determine.
   E None of the above.

   $\sum_{x \in X(\Omega)} \mathbb{P}[X = x] = 1$.
   Same for y.
   Definition of PDF.

5. Suppose $X(\omega_1) = 1, X(\omega_2) = 1, X(\omega_3) = 0$. If $\Omega = \{\omega_1, \omega_2, \omega_3\}$, what is $X(\Omega)$?

   A $\{1\}$ B $\{1,1,0\}$ C $\{0,1\}$ D $\{\omega_1, \omega_2, \omega_3\}$ E None of the above.

   $X(\Omega)$ is the set of all values X can take.

because $A \subseteq B$.

6. Suppose A and B are events such that $A \subseteq B$. What do we know?

   A $\mathbb{P}[A] \le \mathbb{P}[B]$

   B $\mathbb{P}[B] \le \mathbb{P}[A]$

   C We don't have enough information to compare the probabilities of the events.

   D $\mathbb{P}[B] = 1$

   E None of the above.

$$\mathbb{P}[A] = \sum_{w \in A} \mathbb{P}[w] \le \sum_{w \in B} \mathbb{P}[w] = \mathbb{P}[B]$$

7. [**Show Work**] Suppose X_1 and X_2 are independent and uniform on $\{1, 2, 3, 4, 5\}$. What is $\mathbb{P}[X_1 + X_2 > 3]$?

   A $\frac{21}{25}$

   B $\frac{22}{25}$

   C $\frac{23}{25}$

   D $\frac{24}{25}$

   E $\frac{25}{25}$

Consider event $A = X_1 + X_2 \le 3$.

Outcomes in A are $(X_1, X_2) = (1,2), (2,1), (1,1)$

So $\mathbb{P}[A] = \frac{3}{25}$.

$\mathbb{P}[X_1 + X_2 > 3] = 1 - \mathbb{P}[A] = \frac{22}{25}$.

8. Suppose X_1 is the number of heads in n fair coin tosses, and X_2 is 1 if all coins match and 0, otherwise. What do we know about $\mathbb{E}[X_1 + X_2]$?

   ~~A~~ The variables are dependent, so the expected value is unknown.

   ~~B~~ $\mathbb{E}[X_1 + X_2] = \mathbb{E}[X_1] \times \mathbb{E}[X_2]$

   ~~C~~ $\mathbb{E}[X_1 + X_2] = 0$

   D $\mathbb{E}[X_1 + X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2]$

   E None of the above.

Expectation is linear.

9. [**Show Work**] What is $\mathbb{E}[X_2]$ for the random variable X_2 defined in the previous question?

   A 0

   B $\frac{1}{2^n}$

   C $\frac{2}{2^n}$

   D $\frac{1}{2}$

   E 1

$\mathbb{E}[X_2] = \mathbb{P}[X_2 = 1] \times 1 + \mathbb{P}[X_2 = 0] \times 0$

$\mathbb{P}[X_2 = 1] = \frac{2}{2^n}$ (2 outcomes)

10. Suppose I randomly choose 6 different numbers from the set $\{1, \ldots, 10\}$. What is the probability that my set of numbers contains at least two subsets that sum up to the same number?

    A $\frac{1}{\binom{10}{5}}$

    B $1 - \frac{1}{\binom{10}{5}}$

    C 0

    D 1

    E None of the above.

$S = \{n_1, n_2, \ldots, n_6\}$.

There are $2^6 = 64$ subsets. pigeons

There are at most $6 \times 10 = 60$ possible sums. pigeonholes

pigeonhole principle: 63 non-empty subsets, ≤ 60 sums

11. Suppose the answer to the first 5 questions is A, and the answer to the remaining 15 questions is chosen uniformly at random between A-E? How many A answers do you expect to see?

A 5

B 6

C 7

D 8

E 9

Let $X_i = \begin{cases} 1 & \text{if answer to question } i \text{ is A.} \\ 0 & \text{otherwise} \end{cases}$

Then $\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_5] + \mathbb{E}[X_6] + \cdots + \mathbb{E}[X_{20}]$

$1 \qquad 1 \qquad 0.2$

$\mathbb{E}[X] = 5 + 15 \times 0.2 = 8$

12. Suppose the correct answers to all 20 questions are chosen uniformly at random. What is the probability that a letter is never used?

A $\frac{\binom{20}{4}}{\binom{20}{5}}$

B $5 \times \frac{\binom{20}{4}}{\binom{20}{5}}$

C $\frac{4^{20}}{5^{20}}$

D $5 \times \frac{4^{20}}{5^{20}}$

E None of the above.

Exactly 1 letter isn't used.

If more than 1, need also $\mathbb{P}[A,B], \mathbb{P}[A,C], \ldots$ up to all 4-way combination

$\mathbb{P}[A \text{ is never used}] = \left[\frac{4}{5}\right]^{20}$

$\vdots$

$\mathbb{P}[E \text{ is never used}] = \left(\frac{4}{5}\right)^{20}$

$\mathbb{P}[A \text{ is never used} \vee \cdots \vee E \text{ is never used}]$

$= 5 \times \left(\frac{4}{5}\right)^{20}$

13. **[Show Work]** Suppose submitty only allows 1 student to submit at a given time – if multiple students try at the same time, no one succeeds. Suppose all 100 students start at the same time: each student flips a fair coin and if they get a H, they try to submit; otherwise, they wait 1s and try again. How much time do we expect to wait until the first successful submission?

A $\frac{1}{2}$s

B 2s

C $\frac{2^{100}}{100}$s

D $\frac{100}{2^{100}}$s

E None of the above.

$\mathbb{P}[\text{success}] = 100 \times \frac{1}{2^{100}}$

$\binom{100}{1}$

$\mathbb{E}[\text{success}] = \frac{2^{100}}{100}$. A LOT!

14. In Question 13, how much time do we expect to wait until the 3rd successful submission?

A $\frac{3}{2}$s

B 6s

C $\frac{3 \times 2^{100}}{100}$s

D $(\frac{2^{100}}{100} + \frac{2^{99}}{99} + \frac{2^{98}}{98})$s

E None of the above.

$\mathbb{E}[\text{success } 1] = \frac{2^{100}}{100}$

$\mathbb{E}[\text{sucess } 2] = \frac{2^{99}}{99}$ +

$\mathbb{E}[\text{success } 3] = \frac{2^{98}}{98}$

15. **[Show Work]** You flip a fair coin until you get 2 consecutive H. But if you get a T before you get 2 H, you restart. How many flips do you expect to make?

A 4 B 5 C 6 D 7 E None of the above

$\mathbb{E}[X] = \mathbb{E}[X \mid 2H] \cdot \mathbb{P}[2H] + \mathbb{E}[X \mid T] \cdot \mathbb{P}[T]$

$\mathbb{E}[X] = 2 \cdot \frac{1}{4} + (\mathbb{E}(X|T)) \cdot \frac{1}{2} + (2 + \mathbb{E}[X]) \cdot \frac{1}{4} + \mathbb{E}[X \mid HT] \cdot \mathbb{P}[HT]$

$\mathbb{E}[X] = 6$

16. Suppose the weather forecast is correct 95% of the time and it snows on 50% of days. What is the chance of snow if the forecast says it will not snow? Forecast

A 2.5%
B 5%
C 10%
D 20%
E None of the above.

$0 \xrightarrow{0.5} \text{Snow}$: $\xrightarrow{0.95}$ YES, $\xrightarrow{0.05}$ No

$0 \xrightarrow{0.5} \text{No Snow}$: $\xrightarrow{0.05}$ YES, $\xrightarrow{0.95}$ No

$$\mathbb{P}[S \mid N] = \frac{\mathbb{P}[S \wedge N]}{\mathbb{P}[N]} = \frac{0.05 \cdot 0.5}{0.05 \cdot 0.5 + 0.5 \cdot 0.95} = 0.05$$

17. [**Show Work**] Consider the joint PDF of variables X and Y below. What is $\mathbb{E}[XY]$?

A 2
B 1
C $\frac{1}{2}$
D $\frac{5}{12}$
E None of the above.

Only consider non-zero values

$$\mathbb{E}[XY] = (1 \times 1) \cdot \frac{1}{6} + (1 \cdot 3) \cdot \frac{1}{12} = \frac{5}{12}$$

$P_{XY}(x, y)$

$Y \backslash X$	0	1	2	3
0	$\frac{1}{4}$	0	$\frac{1}{8}$	$\frac{1}{8}$
1	$\frac{1}{4}$	$\frac{1}{6}$	0	$\frac{1}{12}$
		$\frac{1}{6}$	$\frac{1}{8}$	$\frac{5}{24}$

row sums

18. [**Show Work**] Consider the joint PDF of variables X and Y in Question 17. What is $\mathbb{E}[X]$?

A $\frac{15}{24}$
B $\frac{25}{24}$
C 1
D 2
E None of the above.

$$\mathbb{E}[X] = \frac{1}{6} \cdot 1 + \frac{1}{8} \cdot 2 + \frac{5}{24} \cdot 3 = \frac{4 + 6 + 15}{24} = \frac{25}{24}$$

19. Consider the joint PDF of variables X and Y in Question 17. How do we show that X and Y are not independent?

~~A Show that $\mathbb{P}[X = x, Y = y] = \mathbb{P}[X = x]\mathbb{P}[Y = y]$ for some x, y.~~
~~B Show that $\mathbb{P}[X = x, Y = y] = \mathbb{P}[X = x]\mathbb{P}[Y = y]$ for all x, y.~~ ← shows they are independent
~~C Show that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.~~
~~D They are independent.~~
E None of the above.

They are dependent because, e.g.,

$$0 = \mathbb{P}[Y = 1, X = 2] \neq \mathbb{P}[Y = 1] \cdot \mathbb{P}[X = 2] = \frac{1}{2} \cdot \frac{1}{6}$$

20. X and Y are independent Binomial random variables over n trials with success probabilities p_1 and p_2, respectively. What is $\mathbb{E}[XY]$?

A n B np_1p_2 C $n^2p_1p_2$ D $n(p_1 + p_2)$ E None of the above

$$\mathbb{E}[XY] \underset{\substack{\uparrow \\ \text{independence}}}{=} \mathbb{E}[X] \cdot \mathbb{E}[Y] = (np_1) \times (n \cdot p_2)$$

Scratch