

Foundations of Computer Science

Lecture 3

Making Precise Statements

Propositions
Compound Propositions and Truth Tables
Predicates and Quantifiers

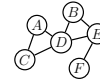


Last Time

1 Sets, $\{3, 5, 11\}$

2 Sequences, 100111001

3 Graphs,



4 Examples of basic proofs.

- ▶ In 4 rounds of group dating, no one meets more than 12 people.
- ▶ x^2 is even “is the same as” x is even.
- ▶ In *any* group of 6 people there is an orgy of 3 mutual friends or a war of 3 mutual enemies.
- ▶ **Axiom:** The Well-Ordering Principle
- ▶ $\sqrt{2}$ is not rational.

Today: Making Precise Statements

1 Making a precise statement: the proposition

2 Complicated precise statements: the compound proposition

- Truth tables

3 Claims about many things

- Predicates
- Quantifiers
- Proofs with quantifiers

Statements can be Ambiguous

1 $2+2=4$.

T

2 $2+2=5$.

F

3 You may have cake OR ice-cream.

(Can you have both?)

4 IF pigs can fly THEN you get an A.

(Pigs can't fly. So, can you get an A?)

5 There is one soulmate for EVERY person.

5a There is a single soul mate that EVERY person shares.

5b EVERY person has their own special soul mate.

Why is ambiguity bad? **Proof!**

We asked questions of our friends to prove 5(b).

Pop Quiz How to prove 5(a)?

A says Sue's their soul mate;
B says Joe's their soul mate;
C says Sue's their soul mate;
D's soul mate is a red Porshe;
E says Sue's their soul mate;
F says Sam's their soul mate.

Propositions are T or F

We use the letters p, q, r, s, \dots to represent propositions.

| | |
|------------------------------|----|
| p : Porky the pig can fly. | F |
| q : You got an A. | T? |
| r : Kilam is an American. | T? |
| s : 4^2 is even. | T |

To get complex statements, combine basic propositions using logical connectors.

Compound Propositions

| | |
|------------------------------|----|
| p : Porky the pig can fly. | F |
| q : You got an A. | T? |
| r : Kilam is an American. | T? |
| s : 4^2 is even. | T |

| Connector | Symbol | An example in words |
|---------------|-------------------|---|
| NOT | $\neg p$ | IT IS NOT THE CASE THAT (Porky the pig can fly) |
| AND | $p \wedge q$ | (Porky the pig can fly) AND (You got an A) |
| OR | $p \vee q$ | (Porky the pig can fly) OR (You got an A) |
| IF... THEN... | $p \rightarrow q$ | IF (Porky the pig can fly) THEN (You got an A) |

Negation (NOT), $\neg p$

The negation $\neg p$ is T when p is F, and the negation $\neg p$ is F when p is T.

“Porky the pig can fly” is F

So,

IT IS NOT THE CASE THAT (Porky the pig can fly) is T

Conjunction (AND), $p \wedge q$

Both p and q must be T for $p \wedge q$ to be T; otherwise $p \wedge q$ is F.

“Porky the pig can fly” is F

We don't know whether “You got an A”.

It does not matter.

(Porky the pig can fly) \wedge (You got an A) is F

Disjunction (OR), $p \vee q$

Both p and q must be F for $p \vee q$ to be F; otherwise $p \vee q$ is T.

“Porky the pig can fly” is F

We don’t know whether “You got an A”.

Now it matters.

(Porky the pig can fly) \vee (You got an A) is T or F

(Depends on whether you got an A.)

Pop Quiz: “You can have cake” OR “You can have ice-cream.” Can you have both?

Truth Tables

| p | q | $\neg p$ | $p \wedge q$ | $p \vee q$ |
|-----|-----|----------|--------------|------------|
| F | F | T | F | F |
| F | T | T | F | T |
| T | F | F | F | T |
| T | T | F | T | T |

The truth table defines the “meaning” of these logical connectors.

Implication (IF... THEN...), $p \rightarrow q$

IF “Porky the pig can fly” **THEN** “You got an A.” (T/F?)
 Suppose T. Since pigs can’t fly, does it mean you can’t get an A?

IF “ n^2 is even”, **THEN** “ n is even.” (T)
 Suppose n^2 is even. Can we conclude $n \neq 5$?

IF “it rained last night” **THEN** “the grass is wet.” (T)
 p : it rained last night
 q : the grass is wet

$$p \rightarrow q$$

What does it *mean* for this common-sense implication to be true?
 What can you conclude? Did it rain last night? Is the grass wet?

Adding New Information to a True Implication: p is T

IF “it rained last night” **THEN** “the grass is wet.”

p : it rained last night
 q : the grass is wet

$$p \rightarrow q$$

Weather report in morning paper: rain last night.

← new information

| | | | |
|---|-------------|-------------------|---|
| IF (it rained last night) THEN (the grass is wet) | T | $p \rightarrow q$ | T |
| It rained last night (from the weather report) | T | p | T |
| Is the grass wet? | YES! | $\therefore q$ | T |

For a **true** implication $p \rightarrow q$, when p is T, you can conclude q is T.

Adding New Information to a True Implication: q is T

IF “it rained last night” THEN “the grass is wet.”

p : it rained last night

q : the grass is wet

$$p \rightarrow q$$

While picking up the morning paper, you see wet grass. ← new information

| | |
|---|-----------------------|
| IF (it rained last night) THEN (the grass is wet) T | $p \rightarrow q$ T |
| The grass is wet (from walking outside) T | q T |
| <hr/> | |
| Did it rain last night? 😞 | $\therefore p$ T or F |

For a **true** implication $p \rightarrow q$, when q is T, you **cannot** conclude p is T.

Adding New Information to a True Implication: p is F

IF “it rained last night” THEN “the grass is wet.”

p : it rained last night

q : the grass is wet

$$p \rightarrow q$$

Weather report in morning paper: no rain last night. ← new information

| | |
|---|-----------------------|
| IF (it rained last night) THEN (the grass is wet) T | $p \rightarrow q$ T |
| It rained last night (from the weather report) F | p F |
| <hr/> | |
| Is the grass wet? 😞 | $\therefore q$ T or F |

For a **true** implication $p \rightarrow q$, when p is F, you **cannot** conclude q is F.

Adding New Information to a True Implication: q is F

IF “it rained last night” THEN “the grass is wet.”

p : it rained last night

q : the grass is wet

$$p \rightarrow q$$

While picking up the paper, you see dry grass. ← new information

| | |
|---|---------------------|
| IF (it rained last night) THEN (the grass is wet) T | $p \rightarrow q$ T |
| It grass is wet (from walking outside) F | q F |
| <hr/> | |
| Did it rain last night? 😞 | $\therefore p$ F |

For a **true** implication $p \rightarrow q$, when q is F, you can conclude p is F.

Implication: Inferences When New Information Comes

For a **true** implication $p \rightarrow q$:

When p is T, you can conclude that q is T.

When q is T, you **cannot** conclude p is T.

When p is F, you **cannot** conclude q is F.

When q is F, you can conclude p is F.

IF $\underbrace{\text{(Porky the pig can fly)}}_F$ THEN $\underbrace{\text{(You got an } A \text{)}}_{\text{can be T or F (phew)}}$

Falsifying “IF (it rained last night) THEN (the grass is wet)”

- You are a scientist collecting data to *verify* that the implication is valid (true).
- One night it rained. That morning the grass was dry.** ← [new information](#)
- What do you think about the implication now?

This is a *falsifying scenario*.

IF (it rains) THEN (the grass is wet) ← not T

$p \rightarrow q$ is F *only* when p is T and q is F. In all other cases $p \rightarrow q$ is T.

Implication is *Extremely* Important, $p \rightarrow q$

All these are $p \rightarrow q$ (p = “it rained last night” and q = “the grass is wet”):

| | |
|--|------------------|
| If it rained last night then the grass is wet. | IF p THEN q |
| It rained last night implies the grass is wet. | p IMPLIES q |
| It rained last night only if the grass is wet. | p ONLY IF q |
| The grass is wet if it rained last night. | q IF p |
| The grass is wet whenever it rains. | q WHENEVER p |

Truth Tables:

| p | q | $\neg p$ | $p \wedge q$ | $p \vee q$ | $p \rightarrow q$ |
|-----|-----|----------|--------------|------------|-------------------|
| F | F | T | F | F | T |
| F | T | T | F | T | T |
| T | F | F | F | T | F |
| T | T | F | T | T | T |

Example: IF (you are hungry OR you are thirsty) THEN you visit the cafeteria

$(p \vee q) \rightarrow r$ where

p : you are hungry
 q : you are thirsty
 r : you visit the cafeteria

- You are thirsty:* q is T. In both cases r is T. (you visit the cafeteria)
- You did visit the cafeteria:* r is T. Are you hungry? We don't know. Are you thirsty? We don't know. (You accompanied your hungry friend (row 2).)
- You did not visit the cafeteria:* r is F. p and q are both F. (You are neither hungry nor thirsty.)

| | p | q | r | $(p \vee q) \rightarrow r$ |
|----|-----|-----|-----|----------------------------|
| 1. | F | F | F | T |
| 2. | F | F | T | T |
| 3. | F | T | F | F |
| 4. | F | T | T | T |
| 5. | T | F | F | F |
| 6. | T | F | T | T |
| 7. | T | T | F | F |
| 8. | T | T | T | T |

Equivalent Compound Statements

| p | q | $p \rightarrow q$ | $\neg q \rightarrow \neg p$ | $\neg p \vee q$ | $q \rightarrow p$ |
|-----|-----|-------------------|-----------------------------|-----------------|-------------------|
| F | F | T | T | T | T |
| F | T | T | T | T | F |
| T | F | F | F | F | T |
| T | T | T | T | T | T |

rains \rightarrow wet grass dry grass \rightarrow no rain no rain \vee wet grass wet grass \rightarrow rain

$$p \rightarrow q \stackrel{\text{equiv}}{=} \neg q \rightarrow \neg p \stackrel{\text{equiv}}{=} \neg p \vee q$$

Order is very important: $p \rightarrow q$ and $q \rightarrow p$ **do not** mean the same thing.

IF I'm dead, THEN my eyes are closed **vs.** IF my eyes are closed, THEN I'm dead

Pop Quiz 3.5. Compound propositions are used for program control flow, especially IF... THEN....

| | | |
|---|--|--|
| if(x > 0 (y > 1 && x < y)) Execute some instructions. | | if(x > 0 y > 1) Execute some instructions. |
|---|--|--|

Use truth-tables to show that both do the same thing. Which do you prefer and why?

Proving an Implication: Reasoning Without Facts

IF (n^2 is even) THEN (n is even).

| | | | |
|-------------------|-----|-----|-------------------|
| | p | q | $p \rightarrow q$ |
| $p : n^2$ is even | F | F | T |
| $q : n$ is even | F | T | T |
| $p \rightarrow q$ | T | F | F |
| | T | T | T |

What is n ? How to prove?

We must show that the highlighted row *cannot* occur.

In this row, q is F: $n = 2k + 1$.

$$n^2 = (2k + 1)^2 = 2(2k^2 + 2k) + 1$$

p *cannot* be T. This row cannot happen: $p \rightarrow q$ is always T. ■

Quantifiers

EVERY person has **A** soulmate.

Kilam has some gray hair.

Everyone has some gray hair.

Any map can be colored with 4 colors with adjacent countries having different colors.

Every even integer $n > 2$ is the sum of 2 primes (*Goldbach, 1742*).

Someone broke this faucet.

There exists a creature with blue eyes and blonde hair.

All cars have four wheels.

These statements are more complex because of *quantifiers*:

EVERY; A; SOME; ANY; ALL; THERE EXISTS.

Compare:

My Ford Escort has four wheels;

ALL cars have four wheels.

Predicates Are Like Functions

ALL cars have four wheels

Define *predicate* $P(c)$ and its *domain*

$C = \{c | c \text{ is a car}\} \leftarrow \text{set of cars}$
 $P(c) = \text{"car } c \text{ has four wheels"}$

"for all c in C , the statement $P(c)$ is true."

$$\forall c \in C : P(c).$$

(\forall means "for all")

| | Predicate | Function |
|---------|--|--|
| Input | $P(c) = \text{"car } c \text{ has four wheels"}$ | $f(x) = x^2$ |
| Output | parameter $c \in C$ statement $P(c)$ | parameter $x \in \mathbb{R}$ value $f(x)$ |
| Example | $P(\text{Jen's VW}) = \text{"car 'Jen's VW' has four wheels"}$ $\forall c \in C : P(c)$ | $f(5) = 25$ $\forall x \in \mathbb{R}, f(x) \geq 0$ |
| Meaning | For all $c \in C$, the statement $P(c)$ is T. | For all $x \in \mathbb{R}$, $f(x)$ is ≥ 0 . |

There EXISTS a Creature with Blue eyes and Blonde Hair

Define *predicate* $Q(a)$ and its *domain*

$A = \{a | a \text{ is a creature}\} \leftarrow \text{set of creatures}$
 $Q(a) = \text{"} a \text{ has blue eyes and blonde hair"}$

"there exists a in A for which the statement $Q(a)$ is true."

$$\exists a \in A : Q(a).$$

(\exists means "there exists")

$G(a) = \text{"} a \text{ has blue eyes"}$
 $H(a) = \text{"} a \text{ has blonde hair"}$

$$\exists a \in A : \underbrace{(G(a) \wedge H(a))}_{\text{compound predicate}}$$

(When the domain is understood, we don't need to keep repeating it. We write $\exists a : Q(a)$, or $\exists a : (G(a) \wedge H(a))$.)

Negating Quantifiers

IT IS NOT THE CASE THAT(There is creature with blue eyes and blonde hair)

Same as: “All creatures don’t have blue eyes and blonde hair”

$$\neg(\exists a \in A : Q(a)) \equiv \forall a \in A : \neg Q(a)$$

IT IS NOT THE CASE THAT(All cars have four wheels)

Same as: “There is a car which does not have four wheels”

$$\neg(\forall c \in C : P(c)) \equiv \exists c \in C : \neg P(c)$$

When you take the negation inside the quantifier and negate the predicate, you must switch quantifiers: $\forall \rightarrow \exists$, $\exists \rightarrow \forall$

There is a Soulmate for Every Person

Define domains and a predicate.

$$A = \{a \mid a \text{ is a person}\}.$$

$$P(a, b) = \text{“Person } a \text{ has as a soul mate person } b\text{.”}$$

- There is some special person b who is a soul mate to every person a .

$$\exists b : (\forall a : P(a, b)).$$

- For every person a , they have their own personal soul mate b .

$$\forall a : (\exists b : P(a, b)).$$

When quantifiers are mixed, the order in which they appear is important for the meaning. Order generally cannot be switched.

Proofs with Quantifiers

Claim 1. $\forall n > 2$: IF n is even, THEN n is a sum of two primes. (*Goldbach, 1742*)

Claim 2. $\exists(a, b, c) \in \mathbb{N}^3 : a^2 + b^2 = c^2$. ($(a, b, c) \in \mathbb{N}^3$ means triples of natural numbers)

Claim 3. $\neg \exists(a, b, c) \in \mathbb{N}^3 : a^3 + b^3 = c^3$.

Claim 4. $\forall(a, b, c) \in \mathbb{N}^3 : a^3 + b^3 \neq c^3$.

Think about what it would take to prove these claims.