

Foundations of Computer Science

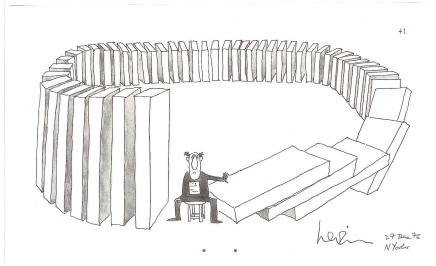
Lecture 6

Strong Induction

Strengthening the Induction Hypothesis

Strong Induction

Many Flavors of Induction



Last Time

- 1 Proving “for all”:
 - ▶ $P(n) : 4^n - 1$ is divisible by 3. $\forall n : P(n)?$
 - ▶ $P(n) : \sum_{i=1}^n i = \frac{1}{2}n(n+1)$. $\forall n : P(n)?$
 - ▶ $P(n) : \sum_{i=1}^n i^2 = \frac{1}{6}n(n+1)(2n+1)$. $\forall n : P(n)?$
- 2 Induction.
- 3 Induction and Well-Ordering.

Today: Twists on Induction

- 1 Solving Harder Problems with Induction
 - $\sum_{i=1}^n \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$
- 2 Strengthening the Induction Hypothesis
 - $n^2 < 2^n$
 - L -tiling.
- 3 Many Flavors of Induction
 - Leaping Induction
 - Postage; $n^3 < 2^n$
 - Strong Induction
 - Fundamental Theorem of Arithmetic
 - Games of Strategy

A Hard Problem: $\sum_{i=1}^n \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$

Proof. $P(n) : \sum_{i=1}^n \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$.

1: **[Base case]** $P(1)$ claims that $1 \leq 2 \times \sqrt{1}$, which is clearly T.

2: **[Induction step]** Show $P(n) \rightarrow P(n+1)$ for all $n \geq 1$ (direct proof)

Assume (induction hypothesis) $P(n)$ is T: $\sum_{i=1}^n \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$.

Show $P(n+1)$ is T: $\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} \leq 2\sqrt{n+1}$.

$$\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} = \sum_{i=1}^n \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n+1}}$$

$$\stackrel{\text{IH}}{\leq} 2\sqrt{n} + \frac{1}{\sqrt{n+1}}$$

$$\stackrel{(\text{lemma})}{\leq} 2\sqrt{n+1}$$

So, $P(n+1)$ is T.

3: By induction, $P(n)$ is T $\forall n \geq 1$. ■

Lemma. $2\sqrt{n} + 1/\sqrt{n+1} \leq 2\sqrt{n+1}$
Proof. By contradiction.
 $2\sqrt{n} + 1/\sqrt{n+1} > 2\sqrt{n+1}$
 $\rightarrow 2\sqrt{n(n+1)} + 1 > 2(n+1)$
 $\rightarrow 4n(n+1) > (2n+1)^2$
 $\rightarrow 0 > 1$ **FISHY!**

Proving Stronger Claims

$$n^2 \leq 2^n \quad \text{for } n \geq 4.$$

Induction Step. Must use $n^2 \leq 2^n$ to show $(n+1)^2 \leq 2^{n+1}$.

$$(n+1)^2 = n^2 + 2n + 1 \leq 2^n + 2n + 1 \stackrel{?}{\leq} 2^n + 2^n = 2^{n+1}$$

What to do with the $2n + 1$?

Would be fine if $2n + 1 \leq 2^n$.

With induction, it can be easier to prove a stronger claim.

Strengthen the Claim: $Q(n)$ Implies $P(n)$

$$Q(n) : (i) n^2 \leq 2^n \quad \text{AND} \quad (ii) 2n + 1 \leq 2^n.$$

$$\boxed{Q(4)} \rightarrow Q(5) \rightarrow Q(6) \rightarrow Q(7) \rightarrow Q(8) \rightarrow Q(9) \rightarrow \dots$$

Proof. $Q(n) : (i) n^2 \leq 2^n \quad \text{AND} \quad (ii) 2n + 1 \leq 2^n$.

1: **[Base case]** $Q(4)$ claims (i) $4^2 \leq 2^4$ AND (ii) $2 \times 4 + 1 \leq 2^4$. Both clearly T.

2: **[Induction step]** Show $Q(n) \rightarrow Q(n+1)$ for $n \geq 4$ (direct proof).

Assume (induction hypothesis) $Q(n)$ is T: (i) $n^2 \leq 2^n$ AND (ii) $2n + 1 \leq 2^n$.

Show $Q(n+1)$ is T: (i) $(n+1)^2 \leq 2^{n+1}$ AND (ii) $2(n+1) + 1 \leq 2^{n+1}$.

$$(i) \quad (n+1)^2 = n^2 + 2n + 1 \leq 2^n + 2^n = 2^{n+1} \quad \checkmark$$

(because from the induction hypothesis $n^2 \leq 2^n$ and $2n + 1 \leq 2^n$)

$$(ii) \quad 2(n+1) + 1 = 2 + 2n + 1 \leq 2^n + 2^n = 2^{n+1} \quad \checkmark$$

(because $2 \leq 2^n$ and from the induction hypothesis $2n + 1 \leq 2^n$)

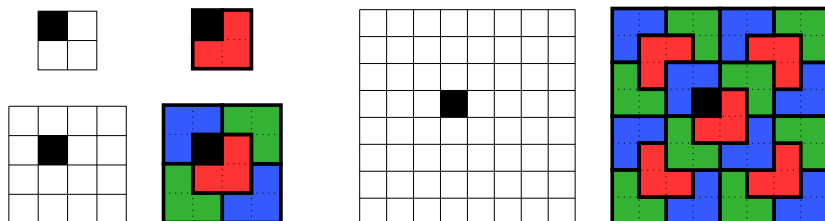
So, $Q(n+1)$ is T.

3: By induction, $Q(n)$ is T $\forall n \geq 4$. ■

L-Tile Land

Can you tile a $2^n \times 2^n$ patio missing a center square. You have only ■ - tiles?

TINKER!

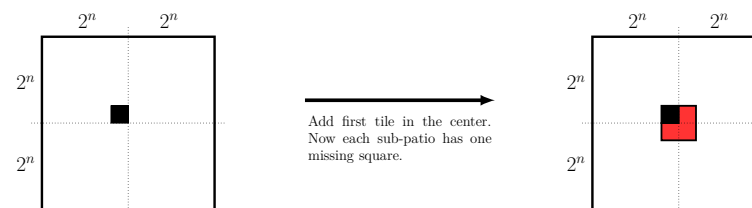


$P(n)$: The $2^n \times 2^n$ grid minus a center-square can be L-tiled.

L-Tile Land: Induction Idea

Suppose $P(n)$ is T. What about $P(n+1)$?

The $2^{n+1} \times 2^{n+1}$ patio can be decomposed into four $2^n \times 2^n$ patios.



Problem. Corner squares are missing. $P(n)$ can be used only if center-square is missing.

Solution. Strengthen claim to also include patios missing corner-squares.

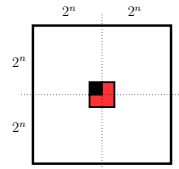
$Q(n)$: (i) The $2^n \times 2^n$ grid missing a **center-square** can be L-tiled; AND
(ii) The $2^n \times 2^n$ grid missing a **corner-square** can be L-tiled.

L-Tile Land: Induction Proof of Stronger Claim

Assume $Q(n)$: (i) The $2^n \times 2^n$ grid missing a **center-square** can be L-tiled; AND
 (ii) The $2^n \times 2^n$ grid missing a **corner-square** can be L-tiled.

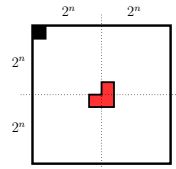
Induction step: Must prove two things for $Q(n+1)$, namely (i) and (ii).

(i) Center square missing.



use $Q(n)$ with corner squares.

(ii) Corner square missing.



use $Q(n)$ with corner squares.

Your task: Add base cases and complete the formal proof.

Exercise 6.4. What if the missing square is some random square? Strengthen further.

A Tricky Induction Problem

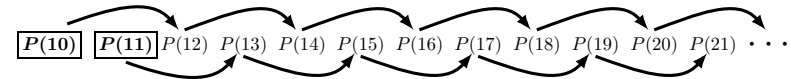
$$P(n) : n^3 < 2^n, \quad \text{for } n \geq 10. \quad (\text{Exercise 6.2})$$

Suppose $P(n)$ is T. Consider $P(n+2) : (n+2)^3 < 2^{n+2}$

$$\begin{aligned} (n+2)^3 &= n^3 + 6n^2 + 12n + 8 \\ &< n^3 + n \cdot n^2 + n^2 \cdot n + n^3 && (n \geq 10 \rightarrow 6 < n; 12 < n^2; 8 < n^3) \\ &= 4n^3 < 4 \cdot 2^n = 2^{n+2} && (P(n) \text{ gives } n^3 < 2^n) \end{aligned}$$

$$P(n) \rightarrow P(n+2).$$

Base cases. $P(10) : 10^3 < 2^{10}$ ✓ and $P(11) : 11^3 < 2^{11}$ ✓

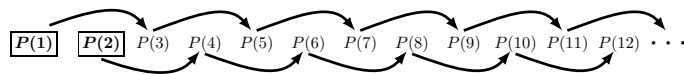


Leaping Induction

Induction. One base case.

$$P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow P(5) \rightarrow \dots$$

Leaping Induction. More than one base case.



Example. Postage greater than 5¢ can be made using 3¢ and 4¢ stamps.

3¢	4¢	5¢	6¢	7¢	8¢	9¢	10¢	11¢	12¢	...
3	4	-	3,3	3,4	4,4	3,3,3	3,3,4	3,4,4	4,4,4	...

$P(n)$: Postage of n cents can be made using only 3¢ and 4¢ stamps.

$$P(n) \rightarrow P(n+3) \quad (\text{add a 3¢ stamp to } n)$$

Base cases: 6¢, 7¢, 8¢.

Practice. Exercise 6.6

Fundamental Theorem of Arithmetic

$$2015 = 5 \times 13 \times 31.$$

Theorem. (The Primes $\mathcal{P} = \{2, 3, 5, 7, 11, \dots\}$ are the atoms for numbers.)

Suppose $n \geq 2$. Then,

- ① n can be written as a product of factors all of which are prime.
- ② The representation of n as a product of primes is unique (up to reordering).

$P(n)$: n is a product of primes.

What's the first thing we do? **TINKER!**

$$2016 = 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 7.$$

Wow! No similarity between the factors of 2015 and those of 2016.

How will $P(n)$ help us to prove $P(n+1)$?

Much “Stronger” Induction Claim

Do smaller values of n help with 2016? Yes!

$$2016 = 32 \times 63$$

$$P(32) \wedge P(63) \rightarrow P(2016) \quad (\text{like leaping induction})$$

Much Stronger Claim:

$Q(n)$: 2, 3, ..., n are all products of primes.

$P(n)$: n is a product of primes. (Compare)

$$Q(n) = P(2) \wedge P(3) \wedge P(4) \wedge \dots \wedge P(n).$$

Surprise! The much stronger claim is *much* easier to prove. Also, $Q(n) \rightarrow P(n)$.

Fundamental Theorem of Arithmetic: Proof of Part (i)

$P(n)$: n is a product of primes.

$$Q(n) = P(2) \wedge P(3) \wedge P(4) \wedge \dots \wedge P(n).$$

Proof. (By Induction that $Q(n)$ is T for $n \geq 2$.)

1: **[Base case]** $Q(1)$ claims that 2 is a product of primes, which is clearly T.

2: **[Induction step]** Show $Q(n) \rightarrow Q(n+1)$ for $n \geq 2$ (direct proof).

Assume $Q(n)$ is T: each of 2, 3, ..., n are a product of primes.

Show $Q(n+1)$ is T: each of 2, 3, ..., $n, n+1$ is a product of primes.

Since we assumed $Q(n)$, we already have that 2, 3, ..., n are products of primes.

To prove $Q(n+1)$, we only need to prove $n+1$ is a product of primes.

- $n+1$ is prime. Done (nothing to prove).

- $n+1$ is not prime, $n+1 = k\ell$, where $2 \leq k, \ell \leq n$.

$P(k) \rightarrow k$ is a product of primes.

$P(\ell) \rightarrow \ell$ is a product of primes.

$n+1 = k\ell$ is a product of primes and $Q(n+1)$ is T.

3: By induction, $Q(n)$ is T $\forall n \geq 2$. ■

Strong Induction

Strong Induction. To prove $P(n) \forall n \geq 1$ by strong induction, you use induction to prove the *stronger* claim:

$Q(n)$: each of $P(1), P(2), \dots, P(n)$ are T.

	Ordinary Induction	Strong Induction
Base Case	Prove $P(1)$	Prove $Q(1) = P(1)$
Induction Step	Assume: $P(n)$ Prove: $P(n+1)$	Assume: $Q(n) = P(1) \wedge P(2) \wedge \dots \wedge P(n)$ Prove: $P(n+1)$

Strong induction is always easier.

Every $n \geq 1$ Has a Binary Expansion

$P(n)$: Every $n \geq 1$ is a sum of distinct powers of two (its binary expansion).

$$22 = 2^1 + 2^2 + 2^4. \quad (22_{\text{binary}} = 1\ 0\ 1\ 1\ 0.)$$

Base Case: $P(1)$ is T: $1 = 2^0$

Strong Induction: Assume $P(1) \wedge P(2) \wedge \dots \wedge P(n)$ and prove $P(n+1)$.

If n is even, then $n+1 = 2^0 +$ binary expansion of n ,

$$\text{e.g. } 23 = 2^0 + \frac{2^1 + 2^2 + 2^4}{22}$$

If n is odd, then multiply each term in the expansion of $\frac{1}{2}(n+1)$ by 2 to get $n+1$.

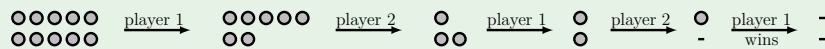
$$\text{e.g. } 24 = 2 \times \frac{(2^2 + 2^3)}{12} = 2^3 + 2^4$$

Exercise. Give the formal proof by strong induction.

The Many Applications of Induction

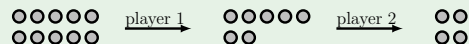
Tournament rankings, greedy or recursive algorithms, **games of strategy**, . . .

Equal Pile Nim (old English/German: to steal or pilfer)



$P(n)$: Player 2 can win the game that starts with n pennies in each row.

Equalization strategy:



Player 2 can always return the game to *smaller* equal piles.

If Player 2 wins the smaller game, Player 2 wins the larger game. That's strong induction!

Exercise. Give the full formal proof by strong induction.

Challenge. What about more than 2 piles. What about unequal piles. (Problem 6.20).

Investigate Further in the Problems

Uniqueness of binary representation as a sum of distinct powers of 2:

Problem 6.27

General Nim:

Problem 6.39

Please, Please, **Please!** Become Good at Induction!

Checklist When Approaching an Induction Problem.

- Are you trying to prove a "For all . . ." claim?
- Identify the claim $P(n)$, especially the parameter n . Here is an example.
Prove: **geometric mean \leq arithmetic mean.** **What is $P(n)$? What is n ?**
 $P(n)$: **geometric mean \leq arithmetic mean for every set of n positive numbers.**
Identifying the right claim is important.
You may fail because you try to prove too much. Your $P(n+1)$ is too heavy a burden. You may fail because you try to prove too *little*. Your $P(n)$ is too weak a support. You must balance the strength of your claim so that the support is just enough for the burden. — G. Polya (paraphrased).
- Tinker. Does the claim hold for small n ($n = 1, 2, 3, \dots$)? These become base cases.
- Tinker. Can you see why (say) $P(5)$ follows from $P(1), P(2), P(3), P(4)$?
This is the crux of induction; to build up from smaller n to a larger n .
- Determine the type of induction: try strong induction first.
- Write out the skeleton of the proof to see exactly what you need to prove.
- Determine and prove the base cases.
- Prove $P(n+1)$ in the induction step. You *must* use the induction hypothesis.