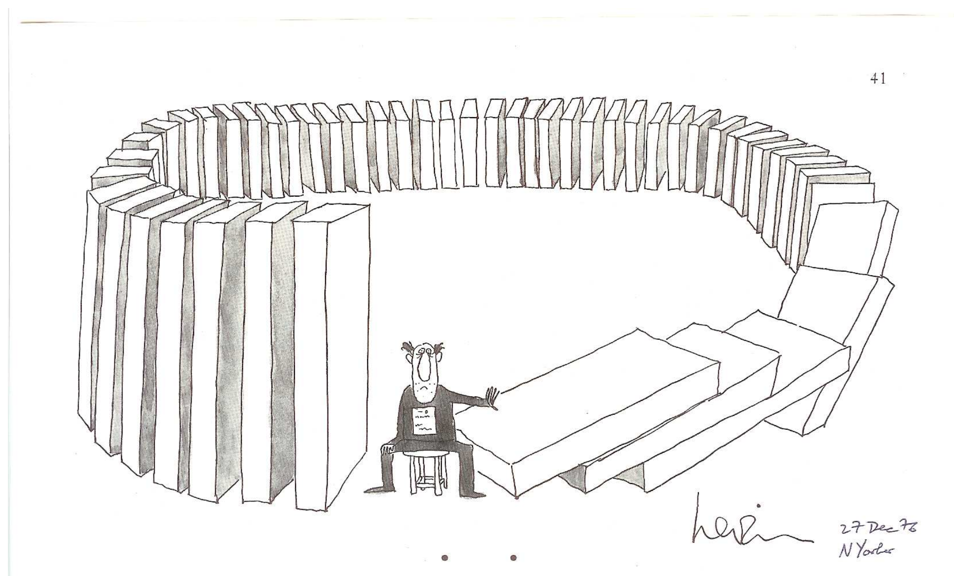
Foundations of Computer Science
Lecture 6

Strong Induction

Strengthening the Induction Hypothesis
Strong Induction
Many Flavors of Induction



Last Time

1. Proving "for all":
   - $P(n) : 4^n - 1$ is divisible by 3. $\forall n : P(n)?$
   - $P(n) : \sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$. $\forall n : P(n)?$
   - $P(n) : \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$. $\forall n : P(n)?$
2. Induction.
3. Induction and Well-Ordering.

Today: Twists on Induction

1. Solving Harder Problems with Induction
   - $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \le 2\sqrt{n}$
2. Strengthening the Induction Hypothesis
   - $n^2 < 2^n$
   - L-tiling.
3. Many Flavors of Induction
   - Leaping Induction
     - Postage; $n^3 < 2^n$
   - Strong Induction
     - Fundamental Theorem of Arithmetic
     - Games of Strategy

A Hard Problem: $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \le 2n$

Proof. $P(n) : \Sigma_{i=1}^{n} \frac{1}{\sqrt{i}} \le 2\sqrt{n}$.

1: **[Base case]** $P(1)$ claims that $1 \le 2 \times \sqrt{1}$, which is clearly T.

2: **[Induction step]** Show $P(n) \rightarrow P(n+1)$ for all $n \ge 1$ (direct proof)

Assume (induction hypothesis) $P(n)$ is T: $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \le 2\sqrt{n}$.

Show $P(n+1)$ is T: $\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} \le 2\sqrt{n+1}$.

$$\begin{aligned}
\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} &= \sum_{i=1}^{n} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n+1}} \\
&\overset{\text{IH}}{\le} 2\sqrt{n} + \frac{1}{\sqrt{n+1}} \\
&\overset{\text{(lemma)}}{\le} 2\sqrt{n+1}
\end{aligned}$$

Lemma. $2\sqrt{n}+1/\sqrt{n+1} \le 2\sqrt{n+1}$

Proof. By contradiction.

$$\begin{aligned}
& 2\sqrt{n} + 1/\sqrt{n+1} > 2\sqrt{n+1} \\
\rightarrow\ & 2\sqrt{n(n+1)} + 1 > 2(n+1) \\
\rightarrow\ & 4n(n+1) > (2n+1)^2 \\
\rightarrow\ & 0 > 1 \quad \textbf{FISHY!}
\end{aligned}$$

So, $P(n+1)$ is T.

3: By induction, $P(n)$ is T $\forall n \ge 1$. ■

Proving Stronger Claims

$$n^2 \le 2^n \qquad \text{for } n \ge 4.$$

Induction Step. Must use $n^2 \le 2^n$ to show $(n+1)^2 \le 2^{n+1}$.

$$(n+1)^2 \;=\; n^2 + 2n + 1 \;\le\; 2^n + 2n + 1 \overset{?}{\le} 2^n + 2^n = 2^{n+1}$$

What to do with the $2n + 1$?

Would be fine if $2n + 1 \le 2^n$.

With induction, it can be easier to prove a stronger claim.

Strengthen the Claim: $Q(n)$ Implies $P(n)$

$$Q(n) : (i)\ n^2 \le 2^n \qquad \text{AND} \qquad (ii)\ 2n+1 \le 2^n.$$

$$\boxed{Q(4)} \to Q(5) \to Q(6) \to Q(7) \to Q(8) \to Q(9) \to \cdots$$

Proof. $Q(n) : (i)\ n^2 \le 2^n \qquad \text{AND} \qquad (ii)\ 2n+1 \le 2^n.$

1: **[Base case]** $Q(4)$ claims $(i)\ 4^2 \le 2^4$ AND $(ii)\ 2 \times 4 + 1 \le 2^4$. Both clearly T.

2: **[Induction step]** Show $Q(n) \to Q(n+1)$ for $n \ge 4$ (direct proof).

Assume (induction hypothesis) $Q(n)$ is T: $(i)\ n^2 \le 2^n$ AND $(ii)\ 2n+1 \le 2^n$.

Show $Q(n+1)$ is T: $(i)\ (n+1)^2 \le 2^{n+1}$ AND $(ii)\ 2(n+1)+1 \le 2^{n+1}$.

$$(i) \qquad (n+1)^2 = n^2 + 2n + 1 \le 2^n + 2^n = 2^{n+1} \ ✓$$

(because from the induction hypothesis $n^2 \le 2^n$ **and** $2n+1 \le 2^n$)

$$(ii) \quad 2(n+1)+1 = 2 + 2n + 1 \le 2^n + 2^n = 2^{n+1} \ ✓$$

(because $2 \le 2^n$ and from the induction hypothesis $2n+1 \le 2^n$)

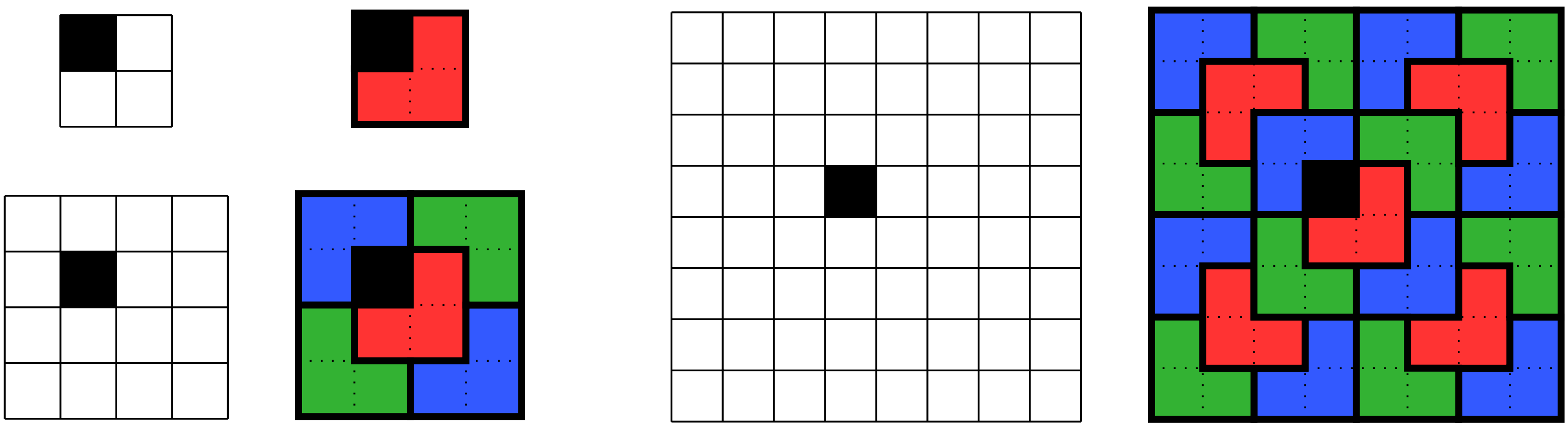
So, $Q(n+1)$ is T.

3: By induction, $Q(n)$ is T $\forall n \ge 4$. ■

L-Tile Land

Can you tile a $2^n \times 2^n$ patio missing a center square. You have only  – tiles?

TINKER!

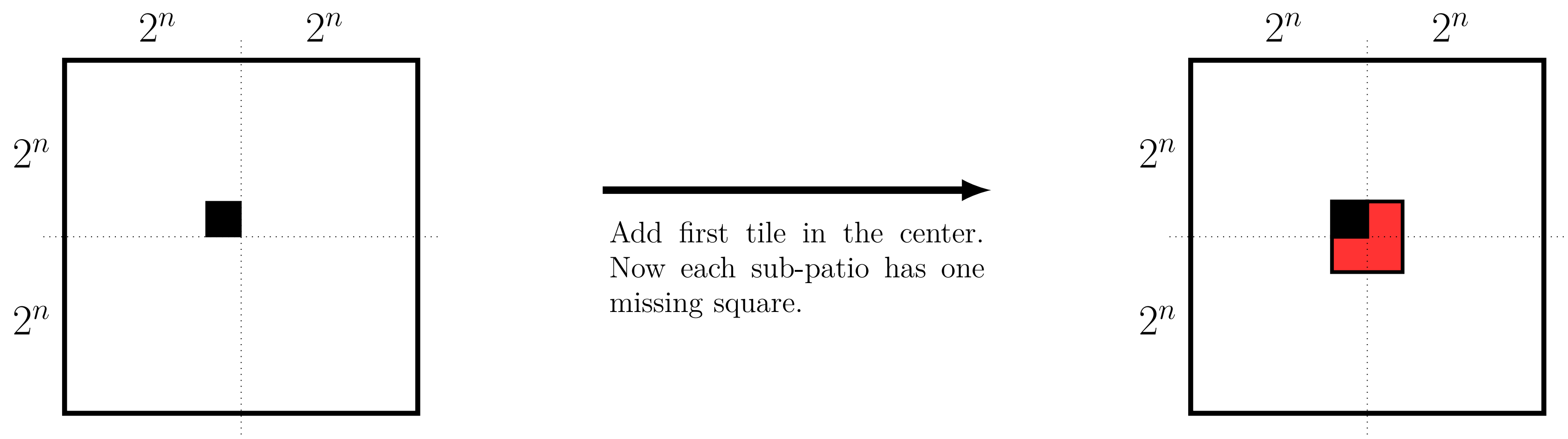


$P(n)$: The $2^n \times 2^n$ grid minus a center-square can be L-tiled.

L-Tile Land: Induction Idea

Suppose $P(n)$ is T. What about $P(n+1)$?

The $2^{n+1} \times 2^{n+1}$ patio can be decomposed into four $2^n \times 2^n$ patios.



Problem. Corner squares are missing. $P(n)$ can be used only if center-square is missing.

Solution. Strengthen claim to also include patios missing corner-squares.

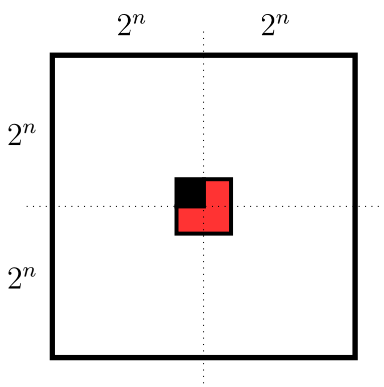
$Q(n)$: (i) The $2^n \times 2^n$ grid missing a **center-square** can be L-tiled; AND
(ii) The $2^n \times 2^n$ grid missing a **corner-square** can be L-tiled.

L-Tile Land: Induction Proof of Stronger Claim

Assume $Q(n)$: (i) The $2^n \times 2^n$ grid missing a **center-square** can be L-tiled; AND
(ii) The $2^n \times 2^n$ grid missing a **corner-square** can be L-tiled.

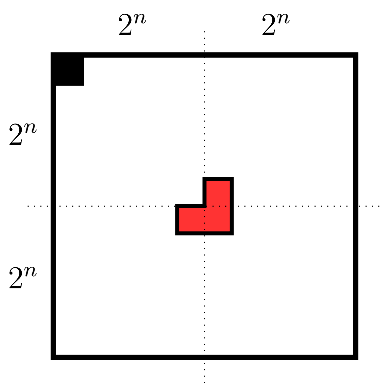
Induction step: Must prove two things for $Q(n+1)$, namely (i) *and* (ii).

(i) Center square missing.



use $Q(n)$ with corner squares.

(ii) Corner square missing.



use $Q(n)$ with corner squares.

Your task: Add base cases and complete the formal proof.

Exercise 6.4. What if the missing square is some random square? Strengthen further.

A Tricky Induction Problem

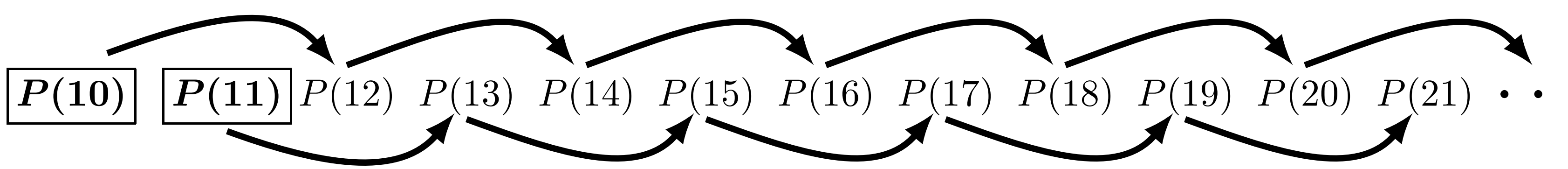
$$P(n) : n^3 < 2^n, \qquad \text{for } n \geq 10. \qquad \text{(Exercise 6.2)}$$

Suppose $P(n)$ is T. Consider $P(n+2) : (n+2)^3 < 2^{n+2}$?

$$\begin{aligned}
(n+2)^3 &= n^3 + \mathbf{6}n^2 + \mathbf{12}n + \mathbf{8} \\
&< n^3 + \boldsymbol{n} \cdot n^2 + \boldsymbol{n^2} \cdot n + \boldsymbol{n^3} && (\boldsymbol{n \geq 10} \rightarrow 6 < n;\ 12 < n^2;\ 8 < n^3) \\
&= 4n^3 \ < \ 4 \cdot 2^n = 2^{n+2} && (P(n) \text{ gives } n^3 < 2^n)
\end{aligned}$$

$$P(n) \rightarrow P(n+2).$$

Base cases. $P(10) : 10^3 < 2^{10}$ ✓ *and* $P(11) : 11^3 < 2^{11}$ ✓

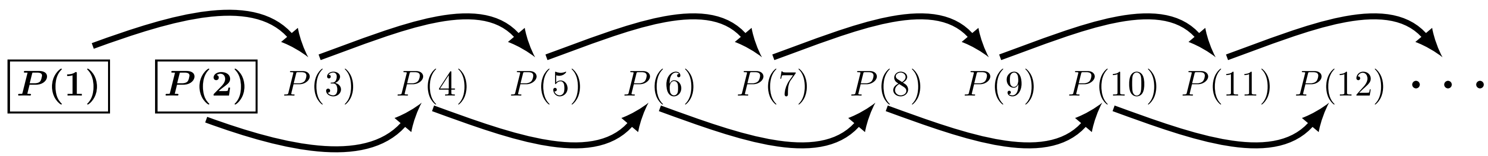


Leaping Induction

Induction. One base case.

$$\boxed{P(1)} \to P(2) \to P(3) \to P(4) \to P(5) \to \cdots$$

Leaping Induction. More than one base case.



Example. Postage greater than 5¢ can be made using 3¢ and 4¢ stamps.

3¢	4¢	5¢	**6¢**	**7¢**	**8¢**	9¢	10¢	11¢	12¢	$\cdots$
3	4	–	**3,3**	**3,4**	**4,4**	3,3,3	3,3,4	3,4,4	4,4,4	$\cdots$

$P(n)$: Postage of n cents can be made using only 3¢ and 4¢ stamps.

$P(n) \to P(n+3)$ (add a 3¢ stamp to n)

Base cases: 6¢, 7¢, 8¢.

Practice. Exercise 6.6

Fundamental Theorem of Arithmetic

$$2015 = 5 \times 13 \times 31.$$

Theorem. (The Primes $\mathcal{P} = \{2, 3, 5, 7, 11, \ldots\}$ are the atoms for numbers.)

Suppose $n \geq 2$. Then,

(i) n **can be written as a product of factors all of which are prime.**

(ii) The representation of n as a product of primes is unique (up to reordering).

$$P(n) : n \text{ is a product of primes.}$$

What's the first thing we do? **TINKER!**

$$2016 = 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 7.$$

Wow! No similarity between the factors of 2015 and those of 2016.

How will $P(n)$ help us to prove $P(n+1)$?

Much "Stronger" Induction Claim

Do smaller values of n help with 2016? Yes!

$$2016 = 32 \times 63$$

$$P(32) \wedge P(63) \rightarrow P(2016)$$ (like leaping induction)

Much Stronger Claim:

$Q(n)$: $2, 3, \ldots, n$ are all products of primes.

$P(n)$: n is a product of primes. (Compare)

$$Q(n) = P(2) \wedge P(3) \wedge P(4) \wedge \cdots \wedge P(n).$$

Surprise! The much stronger claim is *much* easier to prove. Also, $Q(n) \rightarrow P(n)$.

Fundamental Theorem of Arithmetic: Proof of Part (i)

$P(n)$: n is a product of primes.

$$Q(n) = P(2) \wedge P(3) \wedge P(4) \wedge \cdots \wedge P(n).$$

Proof. (By Induction that $Q(n)$ is T for $n \geq 2$.)

1: **[Base case]** $Q(1)$ claims that 2 is a product of primes, which is clearly T.

2: **[Induction step]** Show $Q(n) \to Q(n+1)$ for $n \geq 2$ (direct proof).

Assume $Q(n)$ is T: each of $2, 3, \ldots, n$ are a product of primes.

Show $Q(n+1)$ is T: each of $2, 3, \ldots, n, n+1$ is a product of primes.

Since we assumed $Q(n)$, we already have that $2, 3, \ldots, n$ are products of primes.
To prove $Q(n+1)$, we only need to prove $n+1$ is a product of primes.

- $n+1$ is prime. Done (nothing to prove).
- $n+1$ is not prime, $n+1 = k\ell$, where $2 \leq k, \ell \leq n$.

$$P(k) \to k \text{ is a product of primes.}$$
$$P(\ell) \to \ell \text{ is a product of primes.}$$

$n+1 = k\ell$ is a product of primes and $Q(n+1)$ is T.

3: By induction, $Q(n)$ is T $\forall n \geq 2$. ■

Strong Induction

Strong Induction. To prove $P(n)\ \forall n \geq 1$ by strong induction, you use induction to prove the *stronger* claim:

$$Q(n) : \text{each of } P(1), P(2), \ldots, P(n) \text{ are } \texttt{T}.$$

	Ordinary Induction	**Strong Induction**
Base Case	Prove $P(1)$	Prove $Q(1) = P(1)$
Induction Step	Assume: $P(n)$	Assume: $Q(n) = P(1) \wedge P(2) \wedge \cdots \wedge P(n)$
	Prove: $P(n+1)$	Prove: $P(n+1)$

Strong induction is always easier.

Every $n \geq 1$ Has a Binary Expansion

$P(n)$: Every $n \geq 1$ is a sum of distinct powers of two (its binary expansion).

$$22 = 2^1 + 2^2 + 2^4. \qquad (22_{\text{binary}} = \overset{2^4}{1}\;\overset{2^3}{0}\;\overset{2^2}{1}\;\overset{2^1}{1}\;\overset{2^0}{0}.)$$

Base Case: $P(1)$ is T: $1 = 2^0$

Strong Induction: Assume $P(1) \wedge P(2) \wedge \cdots \wedge P(n)$ and prove $P(n+1)$.

If n is even, then $n + 1 = 2^0 +$ binary expansion of n,

$$\text{e.g. } 23 = 2^0 + \underbrace{2^1 + 2^2 + 2^4}_{22}$$

If n is odd, then multiply each term in the expansion of $\frac{1}{2}(n+1)$ by 2 to get $n + 1$.

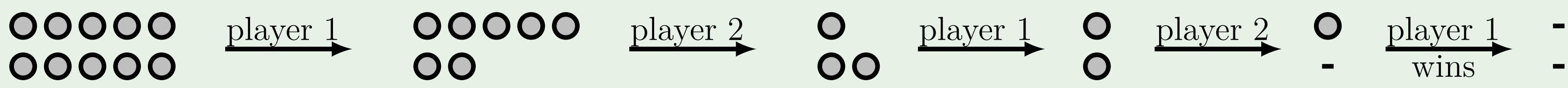
$$\text{e.g. } 24 = 2 \times (\underbrace{2^2 + 2^3}_{12}) = 2^3 + 2^4$$

Exercise. Give the formal proof by strong induction.

The Many Applications of Induction

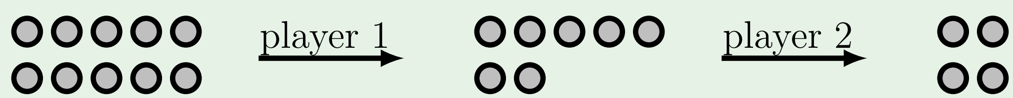
Tournament rankings, greedy or recursive algorithms, **games of strategy**, ...

Equal Pile Nim (old English/German: to steal or pilfer)



$P(n)$: Player 2 can win the game that starts with n pennies in each row.

Equalization strategy:



Player 2 can always return the game to *smaller* equal piles.
If Player 2 wins the smaller game, Player 2 wins the larger game. That's strong induction!

Exercise. Give the full formal proof by strong induction.

Challenge. What about more than 2 piles. What about unequal piles. (Problem 6.20).

Investigate Further in the Problems

Uniqueness of binary representation as a sum of distinct powers of 2:

Problem 6.27

General Nim:

Problem 6.39

Please, Please, **Please**! Become Good at Induction!

Checklist When Approaching an Induction Problem.

- Are you trying to prove a "For all …" claim?
- Identify the claim $P(n)$, especially the parameter n. Here is an example.

  Prove: geometric mean $\leq$ arithmetic mean. What is $P(n)$? What is n?

  $P(n)$: geometric mean $\leq$ arithmetic mean for every set of n positive numbers.

  Identifying the right claim is important.
  You may fail because you try to prove too much. Your $P(n+1)$ is too heavy a burden. You may fail because you try to prove too *little*. Your $P(n)$ is too weak a support. You must balance the strength of your claim so that the support is just enough for the burden. — G. Polya (paraphrased).

- Tinker. Does the claim hold for small n $(n = 1, 2, 3, \ldots)$? These become base cases.
- Tinker. Can you see why (say) $P(5)$ follows from $P(1), P(2), P(3), P(4)$?

  This is the crux of induction; to build up from smaller n to a larger n.

- Determine the type of induction: try strong induction first.
- Write out the skeleton of the proof to see exactly what you need to prove.
- Determine and prove the base cases.
- Prove $P(n + 1)$ in the induction step. You *must* use the induction hypothesis.