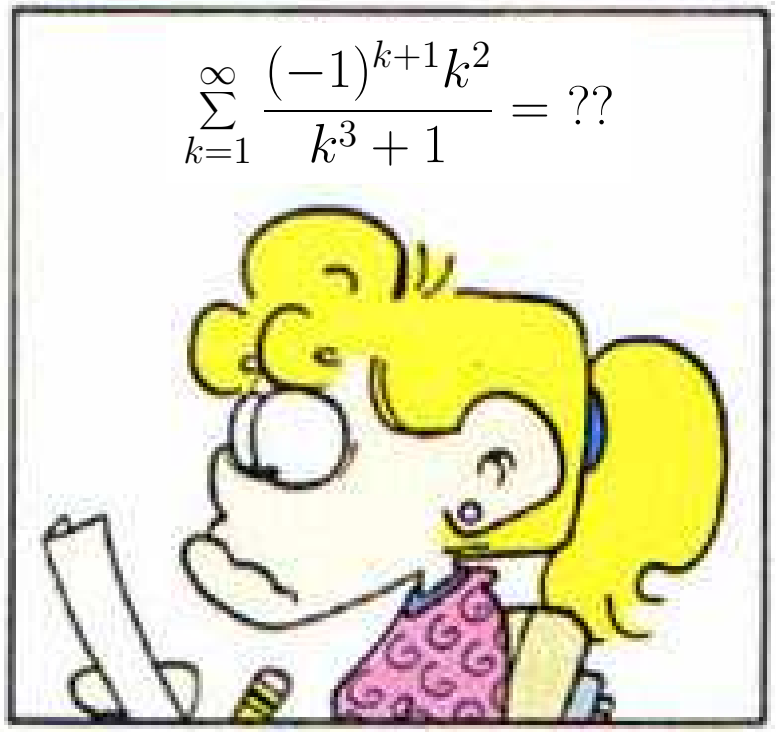
Foundations of Computer Science
Lecture 9

Sums And Asymptotics

Computing Sums
Asymptotics: big-$\Theta(\cdot)$, big-$O(\cdot)$, big-$\Omega(\cdot)$
The Integration Method



Last Time

1. Structural induction: proofs about recursively defined sets.
   - Matched parentheses.
   - $\mathbb{N}$
   - Palindromes.
   - Arithmetic expressions.
   - Rooted Binary Trees (RBT).

Today: Sums And Asymptotics

1. Maximum Substring Sum
2. Computing Sums
3. Asymptotics: Big-Theta, Big-Oh and Big-Omega
4. Integration Method

War Story. A startup asked me about their landing webpage. Users stay 1min, or 2min (about half as many), and so on. To "convert" the 10min user is very different from "converting" the 1 min user. Who should they tailor the webpage to?

Maximum Substring Sum

$$1 \quad -1 \quad -1 \quad 2 \quad 3 \quad 4 \quad -1 \quad -1 \quad 2 \quad 3 \quad -4 \quad 1 \quad 2 \quad -1 \quad -2 \quad 1$$

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More generally, compute the maximum substring sum for

$$a_1 \quad a_2 \quad a_3 \quad a_4 \quad \cdots \quad a_{n-1} \quad a_n$$

(n measures the "size" of the input).

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$$T_1(n) \;=\; 2 + \sum_{i=1}^{n}\Big[2 + \sum_{j=i}^{n}\Big(5 + \sum_{k=i}^{j} 2\Big)\Big].$$

(3 `for` loops)

(What does $\sum_{i=1}^{n}$ mean: Pop Quiz 9.1)

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(3 `for` loops)

$$T_2(n) = 2 + \sum_{i=1}^{n}\left(3 + \sum_{j=i}^{n} 6\right).$$

(2 `for` loops)

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$$T_3(n) = \begin{cases} 3 & n = 1; \\ 2T_3(\frac{1}{2}n) + 6n + 9 & n > 1 \text{ and even}; \\ T(\frac{1}{2}(n+1)) + T(\frac{1}{2}(n-1)) + 6n + 9 & n > 1 \text{ and odd}. \end{cases}$$ (recursive)

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(What does $\sum_{i=1}^{n}$ mean: Pop Quiz 9.1)

Different algorithms have different runtime. Which is best?

Evaluate the Runtimes

$$T_1(n) = 2 + \sum_{i=1}^{n} \Big[2 + \sum_{j=i}^{n} \Big(5 + \sum_{k=i}^{j} 2 \Big) \Big].$$

n	1	2	3	4	5	6	7	8	9	10
$T_1(n)$	11	29	58	100	157	231	324	438	575	737

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T_2 is better than T_1; T_2 versus T_3???

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We need:

1. Simple formulas for $T_1(n), \ldots, T_4(n)$: we need to compute sums and solve recurrences.

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We need:

1. Simple formulas for $T_1(n), \ldots, T_4(n)$: we need to compute sums and solve recurrences.
2. A way to compare runtime-*functions* that captures the essence of the algorithm.

Computing Sums: Tool 1: Constant Rule

$$S_1 = \sum_{i=1}^{10} 3$$

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The *index of summation* is i in these examples.

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Constants (independent of summation index) can be taken outside the sum.

$$S_1 = \sum_{i=1}^{10} 3 = 3 \sum_{i=1}^{10} 1 = 3 \times 10 \qquad S_2 = \sum_{i=1}^{10} j = j \sum_{i=1}^{10} 1 = j \times 10.$$

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Constants (independent of summation index) can be taken outside the sum.

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Pop Quiz 9.2 Compute $T_4(n) = 5 + \sum_{i=1}^{n} 10$.

Computing Sums: Tool 2: Addition Rule

$$S = \sum_{i=1}^{5} (i + i^2)$$

Computing Sums: Tool 2: Addition Rule

$$\begin{aligned} S &= \sum_{i=1}^{5}(i+i^2) \\ &= (1+1^2)+(2+2^2)+(3+3^2)+(4+4^2)+(5+5^2) \end{aligned}$$

Computing Sums: Tool 2: Addition Rule

$$
\begin{aligned}
S &= \sum_{i=1}^{5}(i+i^2) \\
&= (1+1^2)+(2+2^2)+(3+3^2)+(4+4^2)+(5+5^2) \\
&= (1+2+3+4+5)+(1^2+2^2+3^2+4^2+5^2) && \text{(rearrange terms)}
\end{aligned}
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&= \sum_{i=1}^{5} i + \sum_{i=1}^{5} i^2.
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The sum of terms added together is the addition of the individual sums.

$$\sum_i (a(i)+b(i)+c(i)+\cdots) = \sum_i a(i) + \sum_i b(i) + \sum_i c(i) + \cdots$$

Computing Sums: Tool 3: Common Sums

$$\sum_{i=k}^{n} 1 = n+1-k \qquad \sum_{i=1}^{n} i = \tfrac{1}{2}n(n+1) \qquad \sum_{i=0}^{n} 2^i = 2^{n+1}-1$$

$$\sum_{i=1}^{n} f(x) = nf(x) \qquad \sum_{i=1}^{n} i^2 = \tfrac{1}{6}n(n+1)(2n+1) \qquad \sum_{i=0}^{n} \frac{1}{2^i} = 2 - \frac{1}{2^n}$$

$$\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r} \ (r\neq 1) \qquad \sum_{i=1}^{n} i^3 = \tfrac{1}{4}n^2(n+1)^2 \qquad \sum_{i=1}^{n} \log i = \log n!$$

Example: $\sum_{i=1}^{n}(1+2i+2^{i+2})$

$$\sum_{i=1}^{n}(1+2i+2^{i+2}) =$$

Computing Sums: Tool 3: Common Sums

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Example: $\sum_{i=1}^{n}(1+2i+2^{i+2})$

$$\sum_{i=1}^{n}(1+2i+2^{i+2}) = \sum_{i=1}^{n} 1 + \sum_{i=1}^{n} 2i + \sum_{i=1}^{n} 2^{i+2} \qquad \text{(addition rule)}$$

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Example: $\sum_{i=1}^{n}(1+2i+2^{i+2})$

$$\begin{aligned}\sum_{i=1}^{n}(1+2i+2^{i+2}) &= \sum_{i=1}^{n} 1 + \sum_{i=1}^{n} 2i + \sum_{i=1}^{n} 2^{i+2} && \text{(addition rule)}\\ &= \sum_{i=1}^{n} 1 + 2\sum_{i=1}^{n} i + 4\sum_{\boldsymbol{i=1}}^{n} 2^i && \text{(constant rule)}\end{aligned}$$

Computing Sums: Tool 3: Common Sums

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\sum_{i=1}^{n}(1+2i+2^{i+2}) &= \sum_{i=1}^{n} 1 + \sum_{i=1}^{n} 2i + \sum_{i=1}^{n} 2^{i+2} && \text{(addition rule)}\\
&= \sum_{i=1}^{n} 1 + 2\sum_{i=1}^{n} i + 4\sum_{\color{red}{i=1}}^{n} 2^i && \text{(constant rule)}\\
&= n + 2\times\tfrac{1}{2}n(n+1) + 4\cdot(2^{n+1}-1\color{red}{-1}) && \text{(common sums)}
\end{aligned}$$

Computing Sums: Tool 3: Common Sums

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Example: $\sum_{i=1}^{n}(1+2i+2^{i+2})$

$$\begin{aligned}
\sum_{i=1}^{n}(1+2i+2^{i+2}) &= \sum_{i=1}^{n} 1 + \sum_{i=1}^{n} 2i + \sum_{i=1}^{n} 2^{i+2} && \text{(addition rule)}\\
&= \sum_{i=1}^{n} 1 + 2\sum_{i=1}^{n} i + 4\sum_{i=1}^{n} 2^i && \text{(constant rule)}\\
&= n + 2\times\tfrac{1}{2}n(n+1) + 4\cdot(2^{n+1}-1-1) && \text{(common sums)}\\
&= n + n(n+1) + 2^{n+3} - 8 && \text{(common sums)}
\end{aligned}$$

Computing Sums: Tool 3: Nested Sum Rule

$$S_1 = \sum_{i=1}^{3} \sum_{j=1}^{3} 1; \qquad S_2 = \sum_{i=1}^{3} \sum_{j=1}^{i} 1.$$

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To compute a nested sum, start with the innermost sum and proceed outward.

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$$S_1 \qquad \underset{(i=1)}{\sum_{j=1}^{3} 1} \qquad \underset{(i=2)}{\sum_{j=1}^{3} 1} \qquad \underset{(i=3)}{\sum_{j=1}^{3} 1}$$

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$$S_2 \quad \underset{(i=1)}{\sum_{j=1}^{1} 1} \quad \underset{(i=2)}{\sum_{j=1}^{2} 1} \quad \underset{(i=3)}{\sum_{j=1}^{3} 1}$$

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More generally:

$$S(n) = \sum_{i=1}^{n} \sum_{j=1}^{i} 1$$

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More generally:

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Computing a Formula for $T_2(n) = 2 + \sum_{i=1}^{n}\left(3 + \sum_{j=i}^{n} 6\right)$

$$T_2(n) = 2 + \sum_{i=1}^{n}\left(3 + \sum_{j=i}^{n} 6\right) =$$

Computing a Formula for $T_2(n) = 2 + \sum_{i=1}^{n}\left(3 + \sum_{j=i}^{n} 6\right)$

$$T_2(n) = 2 + \sum_{i=1}^{n}\left(3 + \sum_{j=i}^{n} 6\right) = 2 + \sum_{i=1}^{n} 3 + \sum_{i=1}^{n}\sum_{j=i}^{n} 6 \qquad \text{(sum rule)}$$

Computing a Formula for $T_2(n) = 2 + \sum_{i=1}^{n}\left(3 + \sum_{j=i}^{n} 6\right)$

$$
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T_2(n) = 2 + \sum_{i=1}^{n}\left(3 + \sum_{j=i}^{n} 6\right) &= 2 + \sum_{i=1}^{n} 3 + \sum_{i=1}^{n}\sum_{j=i}^{n} 6 && \text{(sum rule)} \\
&= 2 + 3\sum_{i=1}^{n} 1 + \sum_{i=1}^{n}\sum_{j=i}^{n} 6 && \text{(constant rule)}
\end{aligned}
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&= 2 + 3n + \sum_{i=1}^{n}\sum_{j=i}^{n} 6 && \text{(innermost sum)}
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&= 2 + 3n + \sum_{i=1}^{n}\sum_{j=i}^{n} 6 && \text{(innermost sum)} \\
&= 2 + 3n + 6\sum_{i=1}^{n}\sum_{j=i}^{n} 1 && \text{(constant rule)}
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&= 2 + 3n + 6\sum_{i=1}^{n}(n + 1 - i) && \text{(common sum)}
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&= 2 + 3n + 6 \times \tfrac{1}{2}n(n+1) && \text{(common sum)}\\
&= 2 + 6n + 3n^2 && \text{(algebra)}
\end{aligned}
$$

Practice: Compute a Formula for the Sum $\sum_{i=1}^{n} \sum_{j=1}^{i} ij$

$$\sum_{i=1}^{n} \sum_{j=1}^{i} ij \; =$$

Practice: Compute a Formula for the Sum $\sum_{i=1}^{n} \sum_{j=1}^{i} ij$

$$\sum_{i=1}^{n} \sum_{j=1}^{i} ij = \sum_{i=1}^{n} \sum_{j=1}^{i} ij$$ (innermost sum)

Practice: Compute a Formula for the Sum $\sum_{i=1}^{n} \sum_{j=1}^{i} ij$

$$\sum_{i=1}^{n} \sum_{j=1}^{i} ij = \sum_{i=1}^{n} \sum_{j=1}^{i} ij \qquad \text{(innermost sum)}$$

$$= \sum_{i=1}^{n} i \sum_{j=1}^{i} j \qquad \text{(constant rule)}$$

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&= \tfrac{1}{8}n^2(n+1)^2 + \tfrac{1}{12}n(n+1)(2n+1) && \text{(common sums)}\\
&= \tfrac{1}{12}n + \tfrac{3}{8}n^2 + \tfrac{5}{12}n^3 + \tfrac{1}{8}n^4 && \text{(algebra)}
\end{aligned}
$$

Summary of Maximum Substring Sum Algorithms

Runtimes

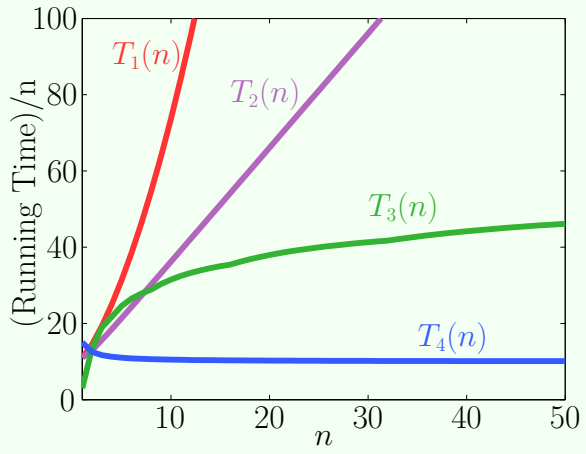
$$\boldsymbol{T_1(n)} = 2 + \tfrac{31}{6}n + \tfrac{7}{2}n^2 + \tfrac{1}{3}n^3$$

$$\boldsymbol{T_2(n)} = 2 + 6n + 3n^2$$

$$3n(\log_2 n + 1) - 9 \leq \boldsymbol{T_3(n)} \leq 12n(\log_2 n + 3) - 9$$

$$\boldsymbol{T_4(n)} = 5 + 10n$$

("simple" formulas for $T_1(n), \ldots, T_4(n)$)



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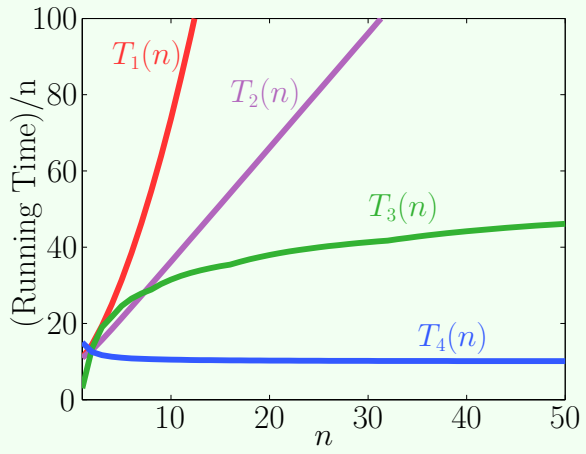

So, which algorithm is best?

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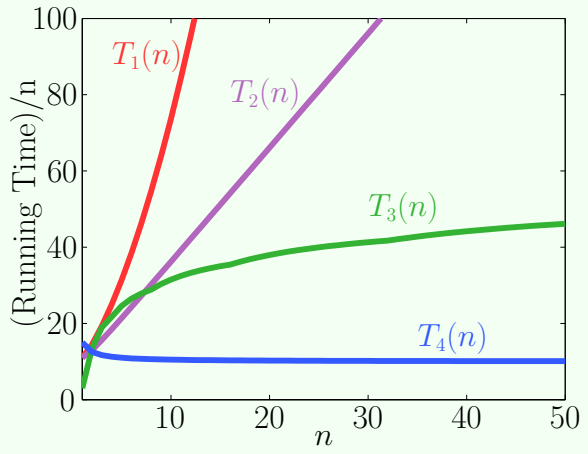

So, which algorithm is best?

Computers solve problems with big inputs. We care about large n.

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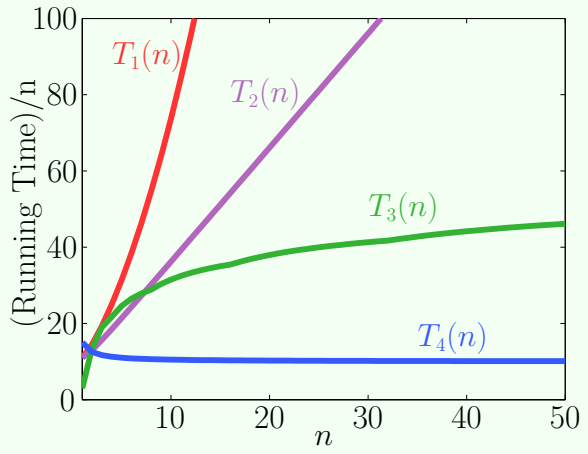

So, which algorithm is best?

Computers solve problems with big inputs. We care about large n.

- Compare runtimes *asymptotically* in the input size n. That is $n \to \infty$.

Summary of Maximum Substring Sum Algorithms

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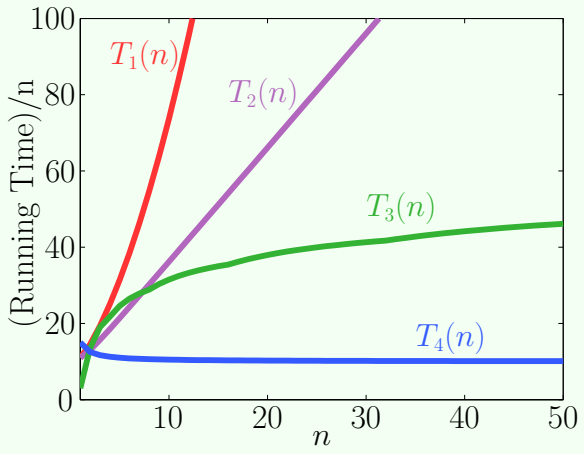

So, which algorithm is best?

Computers solve problems with big inputs. We care about large n.

- Compare runtimes *asymptotically* in the input size n. That is $n \to \infty$.
- Ignore additive and multiplicative constants (minutia). We care about *growth rate*.

Summary of Maximum Substring Sum Algorithms

Runtimes

$\boldsymbol{T_1(n)} = 2 + \frac{31}{6}n + \frac{7}{2}n^2 + \frac{1}{3}n^3$

$\boldsymbol{T_2(n)} = 2 + 6n + 3n^2$

$3n(\log_2 n + 1) - 9 \leq \boldsymbol{T_3(n)} \leq 12n(\log_2 n + 3) - 9$

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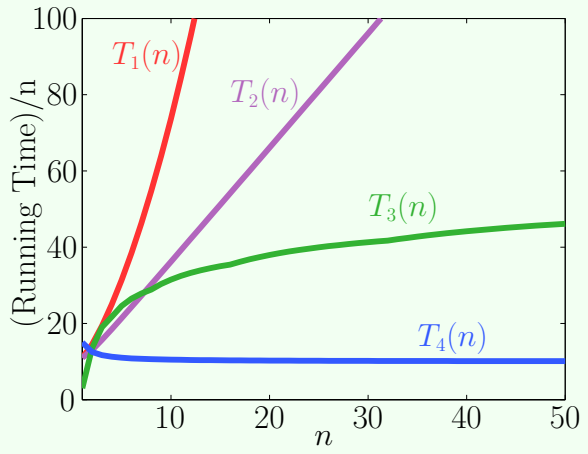

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Algorithm 4 is *linear* in n, $\frac{T_4(n)}{n} \to$ constant.

Asymptotically Linear Functions: $\Theta(n)$, big-Theta-of-n

$T \in \Theta(\boldsymbol{n})$, if there are positive constants c, C for which $c \cdot \boldsymbol{n} \leq T(n) \leq C \cdot \boldsymbol{n}$.

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Linear means in $\Theta(n)$:

$$2n + 7, \qquad 2n + 15\sqrt{n}, \qquad 10^9 n + 3, \qquad 3n + \log n, \qquad 2^{\log_2 n + 4}.$$

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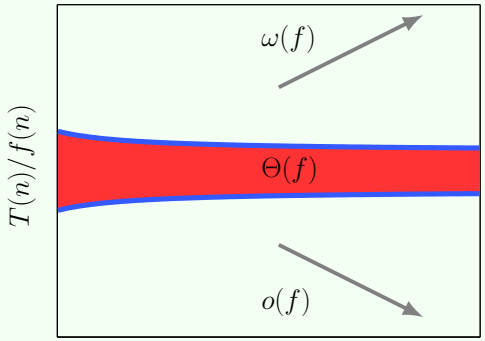
Other runtimes from practice:

log	linear	loglinear	quadratic	cubic	superpolynomial	exponential	factorial	BAD
$\Theta(\log n)$	$\Theta(n)$	$\Theta(n \log n)$	$\Theta(n^2)$	$\Theta(n^3)$	$\Theta(n^{\log n})$	$\Theta(2^n)$	$\Theta(n!)$	$\Theta(n^n)$

General Asymptotics: $\Theta(f)$, big-Theta-of-f

$$\frac{T(n)}{f(n)} \xrightarrow[n\to\infty]{} \begin{cases} \infty & T \in \omega(f),\ \text{“}T < f\text{”}; \\ \text{constant}>0 & T \in \Theta(f),\ \text{“}T = f\text{”}; \\ 0 & T \in o(f),\ \text{“}T < f\text{”}. \end{cases}$$



$T \in o(f)$	$T \in \Theta(f)$	$T \in \omega(f)$
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$$cf(n) \le T(n) \le Cf(n)$$

Examples and Practice. (See also Exercise 9.6)

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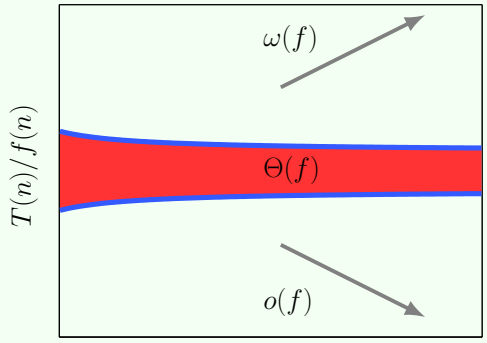

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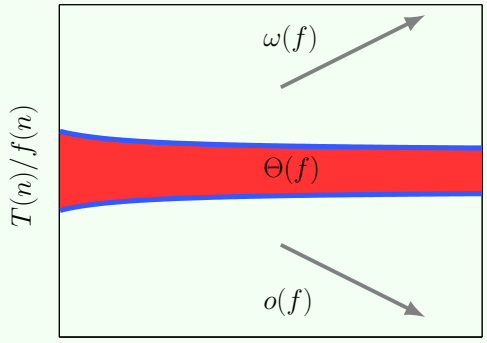

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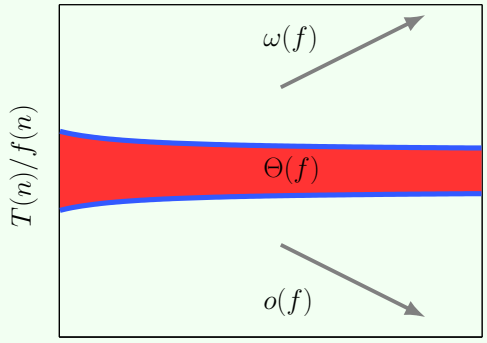

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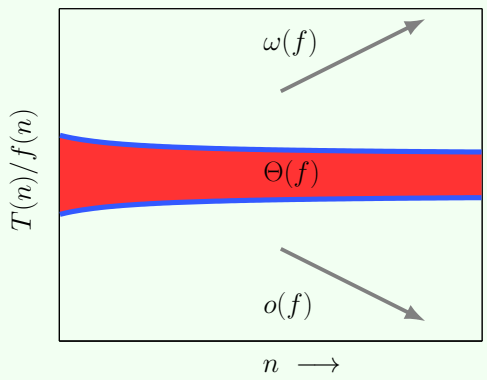

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Examples and Practice. (See also Exercise 9.6)

- For polynomials, growth rate is the highest order.

$2n^2$	$n^2 + n\sqrt{n}$	$n^2 + \log^{256} n$	$n^2 + n^{1.99} \log^{256} n$
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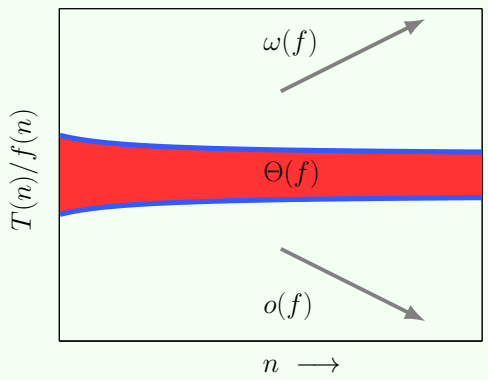

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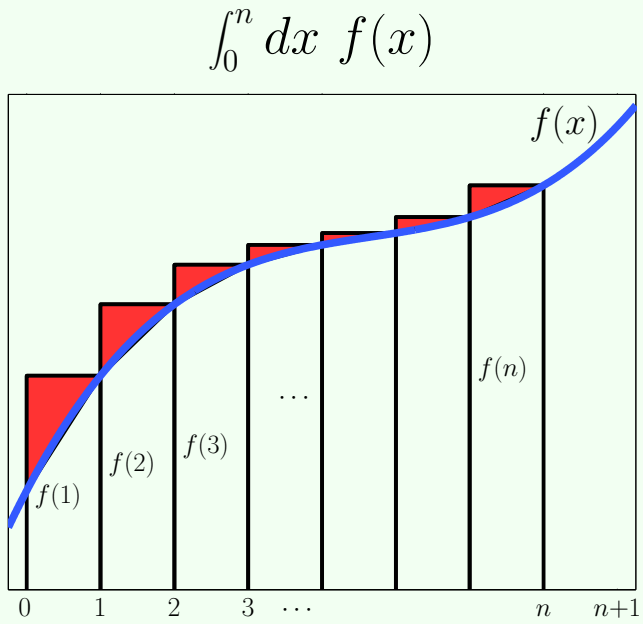
Examples and Practice. (See also Exercise 9.6)

- For polynomials, growth rate is the highest order.
- For nested sums, growth rate is number of nestings plus order of summand.

$2n^2$	$n^2 + n\sqrt{n}$	$n^2 + \log^{256} n$	$n^2 + n^{1.99}\log^{256} n$	$\sum_{i=1}^{n} i$	$\sum_{i=1}^{n}\sum_{j=1}^{i} 1$	$\sum_{i=1}^{n}\sum_{j=1}^{i} ij$
$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^4)$

The Integration Method

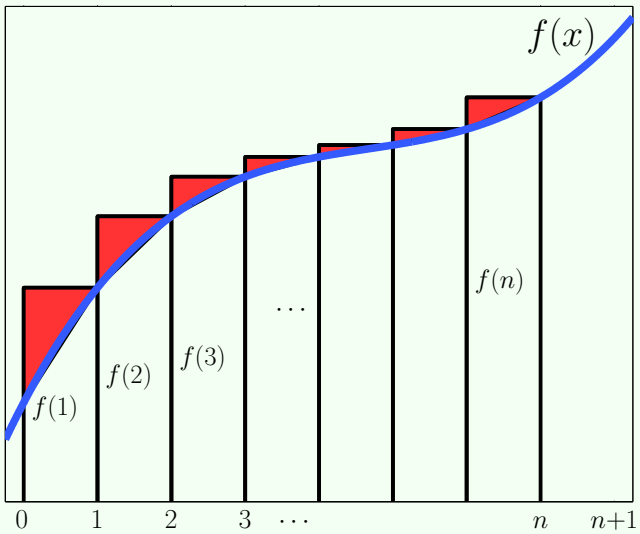


Theorem. (Integration Bound)

For a monotonically increasing function f,

The Integration Method

$$\int_0^n dx\ f(x)$$



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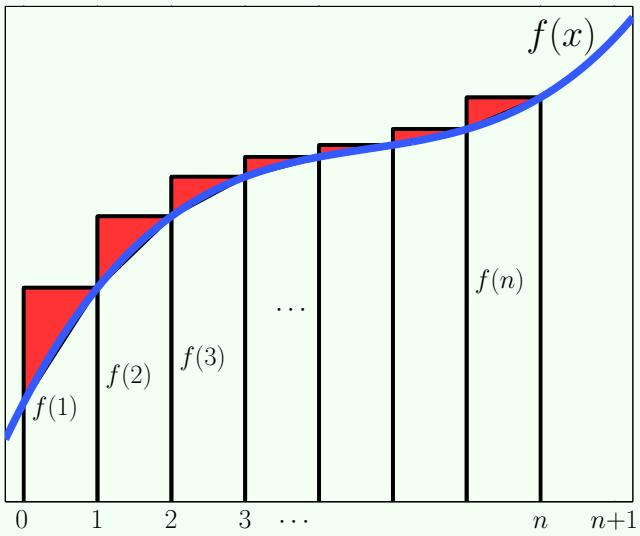
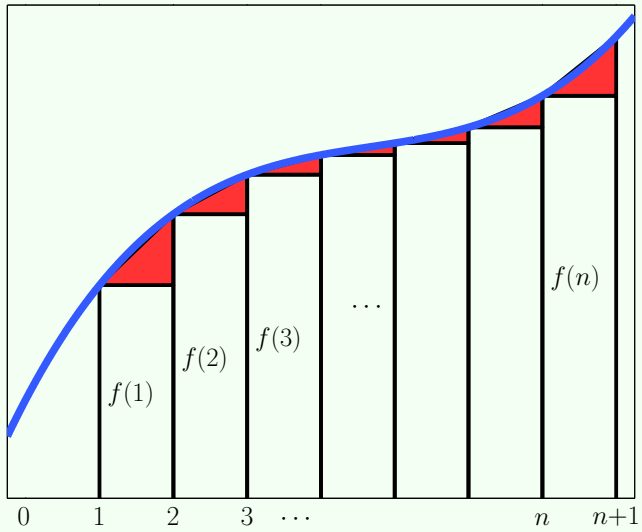


$\int_1^{n+1} dx\ f(x)$



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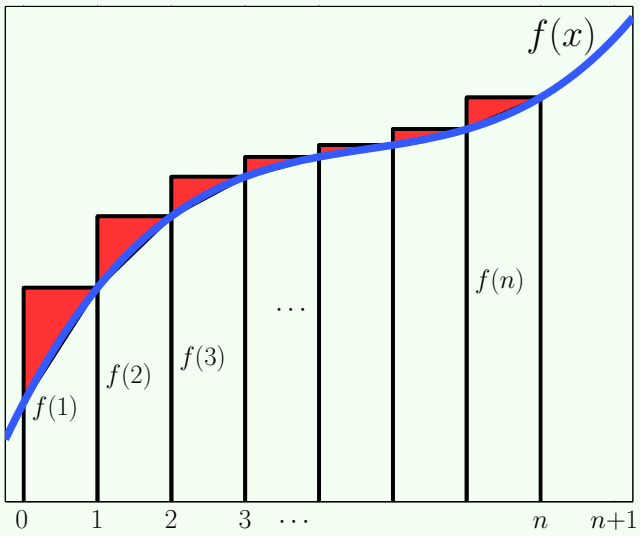

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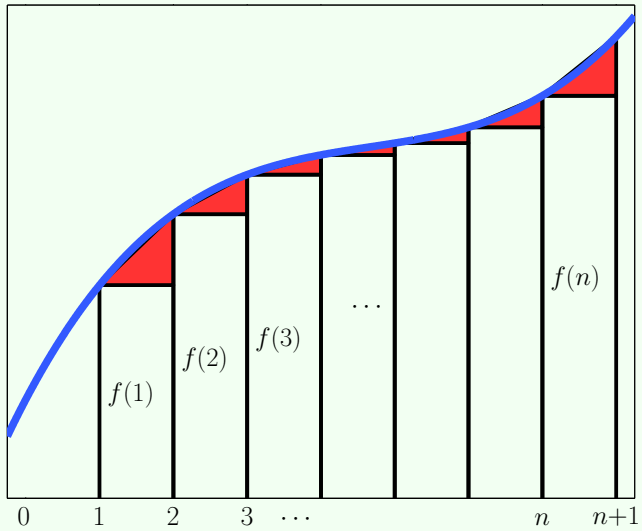

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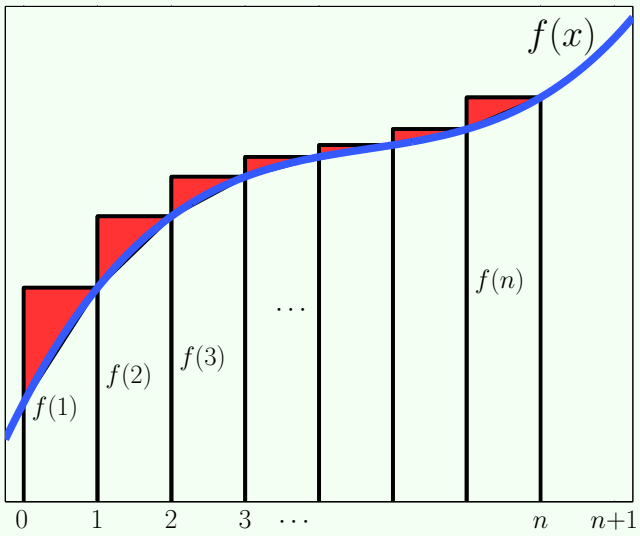

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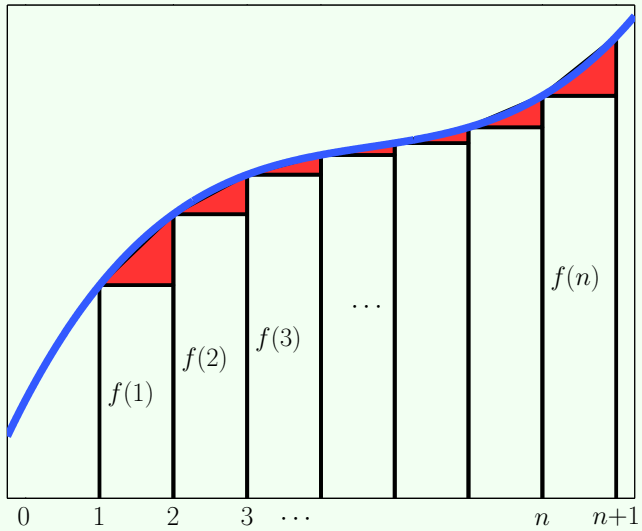

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(If f is monotonically decreasing, the inequalities are reversed.)

Integration For Quickly Getting Asymptotic Behavior

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Integration For Quickly Getting Asymptotic Behavior

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