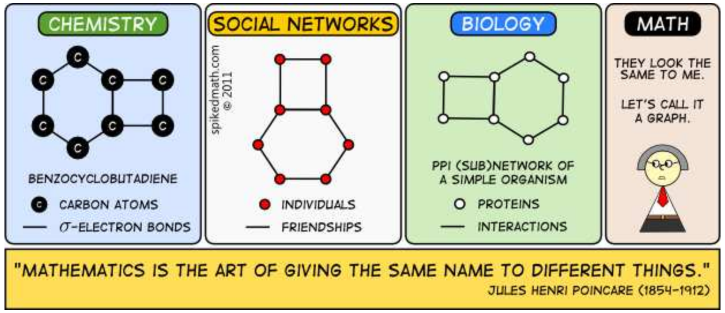
Foundations of Computer Science
Lecture 11

Graphs

Definition and Properties. Equivalence of Graphs.
Degree Sequences and The Handshaking Theorem.
Planar Graphs.
Different Types of Graphs: Multigraph, Weighted, Directed.



Last Time

1. Division, quotient and remainder. Properties of divisibility.
2. Greatest common divisor and Euclid's algorithm.
   - Bezout's Identity: The GCD is the smallest linear combination.
   - Euclid's Lemma: $p|q_1 \cdots q_\ell \to p$ is one of the q_i.
3. Fundamental Theorem of Arithmetic Part II: Uniqeness of prime factorization.
4. Modular arithmetic
   - **Pop Quiz:** What is the last digit of 29^{29}.
5. RSA

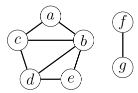
Today: Graphs

1. Graph basics and notation
   - Equivalent graphs: isomorphism
2. Degree sequence
   - Handshaking Theorem
3. Trees
4. Planar graphs
5. Other types of graphs: multigraph, weighted, directed
6. Problem solving with Graphs

Graph Basics and Notation

Graphs model relationships: friendships (e.g. social networks)
connectivity (e.g. cities linked by highways)
conflicts (e.g. radio-stations with listener overlap)

Graph G



Vertices (aka nodes): a b c d e f g $V = \{a, b, c, d, e, f, g\}$.

Edges: a–b, a–c, b–c, b–d, b–e, c–d, d–e, f–g $E = \begin{Bmatrix}(a,b),(a,c),(b,c),(b,d),\\(b,e),(c,d),(d,e),(f,g)\end{Bmatrix}$.

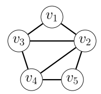
Degree: Number of relationships *e.g.*, degree$(b) = 4$.

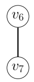
Path: a–c–b–e–d–b $p = acbedb$.

Graph Isomorphism. Relabeling the nodes in G to $v_1, \ldots, v_7$.

$a \to v_1$,
$b \to v_2$,
$c \to v_3$,
$d \to v_4$,
$e \to v_5$,
$f \to v_6$,
$g \to v_7$

Relabeling of Graph G





$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$.

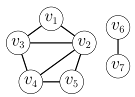
$E = \begin{Bmatrix}(v_1,v_2),(v_1,v_3),(v_2,v_3),(v_2,v_4),\\(v_2,v_5),(v_3,v_4),(v_4,v_5),(v_6,v_7)\end{Bmatrix}$.

If two graphs can be relabeled with $v_1, \ldots, v_n$, giving the *same* edge set, they are equivalent – *isomorphic.*

Practice. Pop Quiz 11.1; Exercise 11.2.

Paths and Connectivity

Graph, G

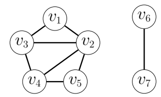


- A *path* from v_1 to v_2 is a sequence of vertices (start is v_1 and end is v_2): $v_1v_3v_2v_5v_4v_2$
- There is an edge in the graph between consecutive vertices in the path.

  v_1 and v_2 are *connected.*
- The length of a path is the number of edges traversed (5).
- *Cycle:* path that starts and ends at a vertex without repeating any edge: $v_1v_2v_3v_1$
- v_1 and v_6 are not connected by a path.
- The graph G is *not* connected (*every* pair of vertices must be connected by a path).
- How can we make G connected?

Graph Representation

Graph



Adjacency List

v_1: v_2, v_3
v_2: v_1, v_3, v_4, v_5
v_3: v_1, v_2, v_4
v_4: v_2, v_3, v_5
v_5: v_2, v_4
v_6: v_7
v_7: v_6

Adjacency Matrix

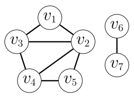
	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1	0	1	1	0	0	0	0
v_2	1	0	1	1	1	0	0
v_3	1	1	0	1	0	0	0
v_4	0	1	1	0	1	0	0
v_5	0	1	0	1	0	0	0
v_6	0	0	0	0	0	0	1
v_7	0	0	0	0	0	1	0

More wasted memory; faster algorithms.

Small redundancy: every edge is "represented" twice.

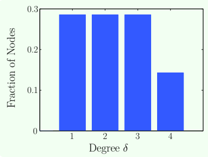
Degree Sequence

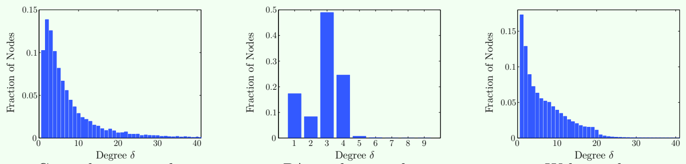
Graph



$$\text{degree } \delta_i = \text{number of } v_i\text{'s neighbors} = \sum_{j=1}^{n} \mathrm{A}_{ij}.$$

	v_2	v_3	v_4	v_1	v_5	v_6	v_7
$\boldsymbol{\delta}$ =	4	3	3	2	2	1	1





Co-author network PA road network Web graph

Complete, K_5 — [4, 4, 4, 4, 4]
Bipartite, $K_{3,2}$ — [3, 3, 2, 2, 2]
Line, L_5 — [2, 2, 2, 1, 1]
Cycle, C_5 — [2, 2, 2, 2, 2]
Star, S_6 — [5, 1, 1, 1, 1, 1]
Wheel, W_6 — [5, 3, 3, 3, 3, 3]

Handshaking Theorem

Pop Quiz. Construct a graph with degree sequence $\boldsymbol{\delta} = [3, 3, 3, 2, 1, 1]$.

Theorem. Handshaking Theorem

For any graph the sum of vertex-degrees equals twice the number of edges, $\sum_{i=1}^{n} \delta_i = 2|E|$.

Proof. Every edge contributes 2 to the sum of degrees. (Why?)
If there are $|E|$ edges, their contribution to the sum of degrees is $2|E|$. ∎

Exercise. Give a formal proof by induction on the number of edges in the graph.

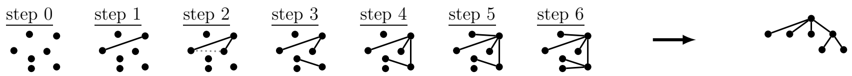
Pop Quiz (Answer). Can't be done: sum of degrees is $3+3+3+2+1+1 = 13$ (odd).

Exercise. At a party a person is odd if they shake hands with an odd number of people.
Show that the number of odd people is even.

Trees (More General than RBTs)

Definition: General Tree.
A tree is a *connected* graph with no cycles.

Building a tree, one edge at a time.

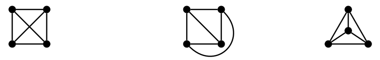


Exercise 11.6. Every tree with n vertices has $n-1$ edges. (We proved this for RBTs.)

Planar Graphs

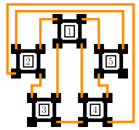
A graph is planar if you can draw it without edge crossings.

Complete graph K_4:

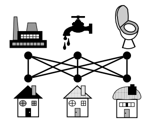


non planar drawing

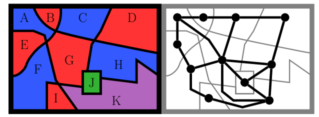
planar drawings → K_4 is planar



Chip design: CPUs must be connected without wire-crossings.



Town planing: connect utilities to homes without pipe-crossings.

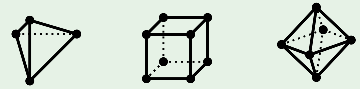


Map coloring: adjacent countries sharing a border must have different colors. The map corresponds to a planar graph.

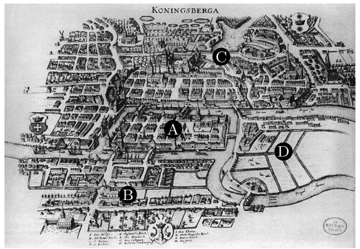
Exercise 11.7. Euler's Invariant Characteristic: $F+V-E=2$.
(Faces, F: outer region or region enclosed by a cycle.)

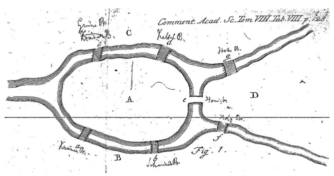
	V	E	F	$F+V-E$
planar K_4	4	6	4	$4+4-6=2$ ✓
planar map	11	17	8	$8+11-17=2$ ✓
pyramid	4	6	4	$4+4-6=2$ ✓
cube	8	12	6	$6+8-12=2$ ✓
octohedron	6	12	8	$8+6-12=2$ ✓

Pyramid Cube Octohedron

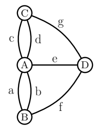


Other Types of Graphs: Multigraph, Weighted, Directed

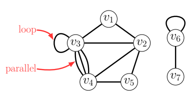




Euler's Multigraph



Multigraph (NOT simple)

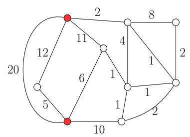


$V=\{v_1,v_2,v_3,v_4,v_5,v_6,v_7\}$.

$$E=\left\{\begin{matrix}(v_1,v_2),(v_1,v_3),(v_2,v_3),(v_2,v_4),\\(v_2,v_5),(v_3,v_4),(v_3,v_4),(v_3,v_4),\\(v_4,v_5),(v_6,v_7),(v_3,v_3),(v_6,v_6)\end{matrix}\right\}$$

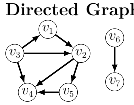
Handshaking Theorem still valid.

Weighted



How quickly can one route between the red ISPs?

Directed Graphs



$V=\{v_1,v_2,v_3,v_4,v_5,v_6,v_7\}$.

$$E=\left\{\begin{matrix}(v_1\to v_2),(v_3\to v_1),(v_3\to v_2),(v_2\to v_4),\\(v_2\to v_5),(v_3\to v_4),(v_5\to v_4),(v_6\to v_7)\end{matrix}\right\}.$$

Ancestry graphs, tournaments, one-way streets, partialy ordered sets (Example 11.6), ...

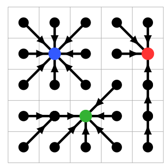
Problem Solving with Graphs

Graphs are everywhere because relationships are everywhere.

On the right is elevation data in a park.
One unit of rain falls on each grid-square.
Water flows to a neighbor of lowest elevation (e.g. 17 → 1)
Where should we install drains and what should their capacities be?

3	2	17	11	12
4	1	18	10	7
21	22	23	16	8
20	13	5	19	9
25	24	6	14	15

Model the problem as a directed graph.
Directed edges indicate how water flows: three disjoint trees.
The red, green and blue vertices are "sinks" (no out-going arrow).



Place drains at the sinks.
Drain capacities: blue=9 units, red=7 units and green=9 units.

The solution pops out once we formulate the problem as a graph.