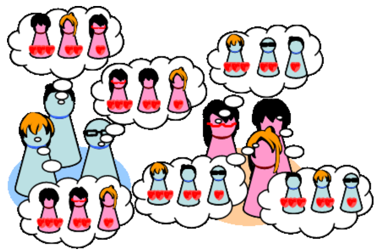
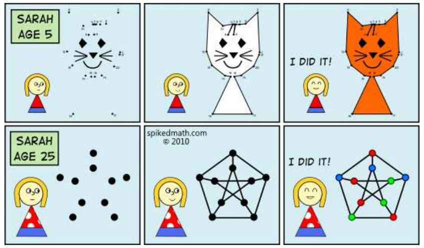
Foundations of Computer Science Lecture 12

Matching and Coloring

Bipartite Graphs and Matching
Stable Marriage
Conflict Graphs and Coloring
Other Graph Problems.





Last Time

1. What is a graph?
2. Equivalent graphs: graph isomorphism.
3. Notation: path, degree, cycle,
4. Some common graphs: $K_n, K_{n,m}, C_n, L_n$.
5. The Handshaking Theorem: $\sum_{i=1}^{n} \delta_i = 2|E|$.
6. Different kinds of graphs: trees; planar; multigraphs; weighted; directed.
7. Problem solving with graphs: first pose the problem as a graph.

Today: Matching and Coloring

1. Matching.
   - Sex in America.
   - Bipartite graphs.
   - Stable marriage: the mathematics of dating.
2. Coloring.
   - Conflict graphs.
3. Other graph problems.
   - Connected components, spaning tree, Euler cycle, newtork flow. (EASY)
   - Hamiltonian cycle, facility location, vertex cover, dominating set. (HARD)

Sex In America: Watchout for Headline News

- The *The Social Organization of Sexuality* (1994) data showed:

  "Men have 74% more opposite-gender partners than women have."

- "*groundbreaking* ABC News 'Primetime Live' survey finds a range of eye-popping sexual activities, fantasies and attitudes in this country, confirming some conventional wisdom, exploding some myths – and *venturing where few scientific surveys have gone before.*"

  "Men have on average 20 sex-partners and women 6 (233% more for men)."

  Claimed margin of error: **2.5%**!

- Not to be outdone, N.Y. Times reported in 2007 on an NIH study:

  "Men have on average 7 partners and women 4 (75% more for men)."

Which survey is right?
Mathematicians should stick to numbers. Real people run the world. No, No, No!

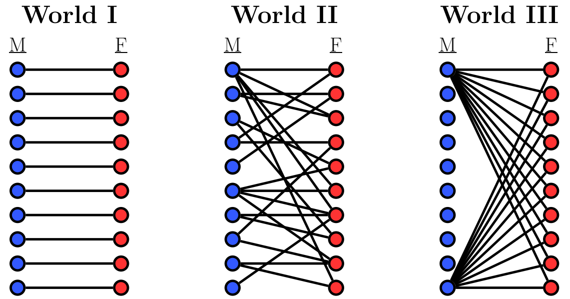
Contentious sensational issues are exactly where mathematics is needed. We *must* face them head on with cold reason instead of flaring emotions.

Sex In America: Resolving the Issue

Modeling asumptions:

- # men = # women.
- All partners are opposite-sex.

← Can be relaxed, Exercise 12.1



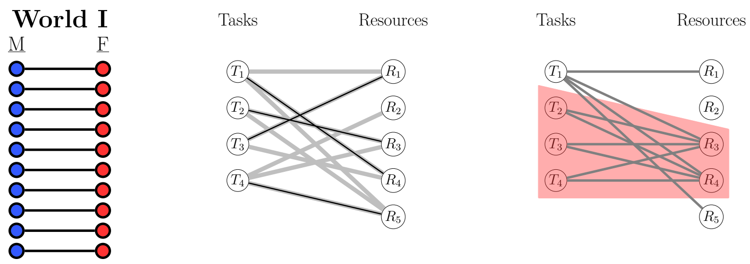
Which world does the media portray?

Theorem. Men and women have the *same* number of partners on average.

Proof. Each edge adds 1 to total partners of male and female → totals are equal → averages are equal. ■

SOS: "Now, there is a basic adding up constraint that these gender differences seem to violate. Logically, men should have the same number of female sex partners as women have male sex partners. We note that this inconsistency has been found, as well, in several other surveys in recent years in the United States, the United Kingdom, France, Finland and elsewhere. The inconsistency constitutes an important puzzle for which we, like others, have no good answer."

Bipartite Matching and Hall's Matching Condition



matching | matching | no matching

Hall's Theorem.
Suppose that for *all* left-subsets X, $|X| \leq |N(X)|$ (Hall's "matching condition"). Then, there is a matching which covers every left-vertex.

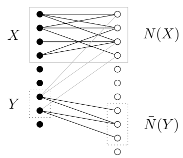
Hall's Theorem says that the obvious necessary condition is also sufficient.

Proof of Hall's Theorem: Induction Step

If for *all* left-subsets X, $|X| \leq |N(X)|$, then there is a left-matching.

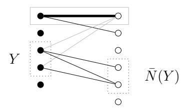
__Case 1.__ There is a proper left-subset X, with $1 \leq |X| \leq n$, for which $|X| = |N(X)|$.

- X has a matching into $N(X)$.
- For any left-subset $Y \subset \overline{X}$, by the matching condition,
  $$|N(X)| + |\bar{N}(Y)| = |N(X \cup Y)| \geq |X \cup Y| = |X| + |Y|$$
  $$\rightarrow\ |\bar{N}(Y)| \geq |Y|.$$
- $\bar{X}$ has as separate matching into $\bar{N}(X)$.



__Case 2.__ For *every* proper left-subset X (with $1 \leq |X| \leq n$), $|X| < |N(X)|$.

- Match the first left-vertex along any edge to a neighbor.
- For any left-subset Y in the remaining graph of n left-vertices,
  $$|\bar{N}(Y)| \geq |N(Y)| - 1 \geq |Y| + 1 - 1 = |Y|$$
- The remaining left-vertices have a matching to the remaining right-vertices.



In both cases, there is a left-matching which covers the $n + 1$ left-vertices. ■

Exercise. If (min left-degree) ≥ (max right-degree) then Hall's condition holds.

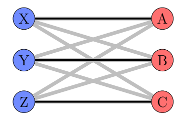
Example 12.3. Building Latin Squares.

Stable Marriage: Mathematics of Dating

Matching with *preferences*: **A**lice, **B**arb and **C**arla want to date **X**avier, **Y**ariv and **Z**ach.

	X	**Y**	**Z**
1.	A	A	B
2.	B	C	A
3.	C	B	C

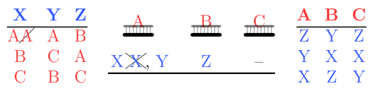
	A	**B**	**C**
1.	Z	Y	Z
2.	Y	X	X
3.	X	Z	Y



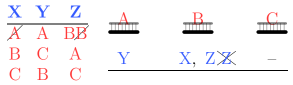
- Yariv prefers Alice to Barb (Barb is Yariv's current mate).
- Alice prefers Yariv to Xavier (Xavier is Alice's current mate).
- Yariv and Alice both prefer each other to their current mates.
- That kind of *volatile* match-up leads to scandal.

A Dating Ritual That Ends with No Volatile Pairs

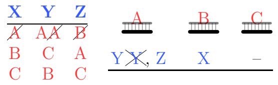
Day 1. Ladies on balconies.
Each gent serenades their top choice.
Ladies ask only their favored suitor to come back ("dating").
A rejected gent will never again woo that lady!



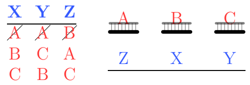
Day 2. Y and Z return to A and B respectively. ("dating")
X goes to B's balcony.
B rejects Z. Z erases B; X and Y will return.



Day 3. Y and X return to A and B respectively. ("dating")
Z goes to A's balcony.
A's patience is rewarded. A rejects Y for her top-choice Z.



Day 4. Each girl has found her boy.
The dating ritual ends with non-volatile marriages
A—Z B—X C—Y

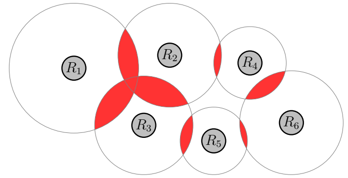


Theorem. [Gale-Shapely, 1962]

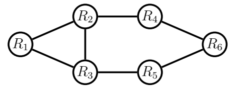
1. For n men and women, the dating ritual ends after at most n^2 days of dating.
2. Every man and woman will be matched at the end.
3. The resulting set of marriages is stable (no volatile pairs).
4. The girls are in control but they lose. ☹

Conflict Graphs and Coloring

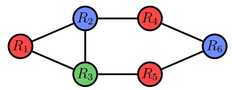
Task 1: Assigning Radio Frequencies



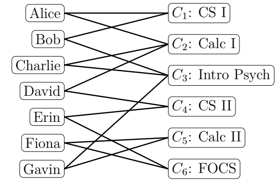
Stations broadcasting to the same listener (red areas) need different frequencies (conflict).



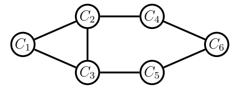
All listeners should be able to listen to all stations.
How many frequencies do you *need*?



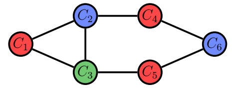
Task 2: Scheduling Course Exams



Courses with the same student need different exam-time (conflict) – Alice causes CS I and Calc I to conflict.



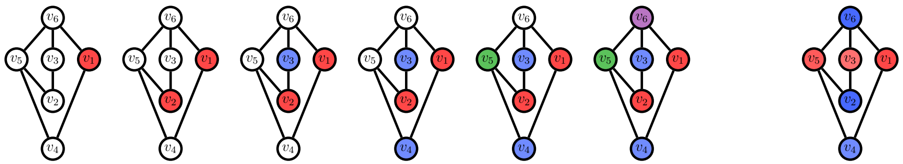
All students need to take all their exams.
How many exam slots do you *need*.



Sequential Greedy Coloring

1: Colors $\{1, 2, 3, \ldots\}$.
2: Let color$(v_1) = 1$.
3: Assume that vertices $v_1, \ldots, v_i$ have been colored. Color v_{i+1} with the *smallest* color so that it does not conflict with any previously colored vertex.

For visual effect, pick the colors $\{1, 2, 3, 4\}$ as {●, ●, ●, ●}.



(2 colors suffice)

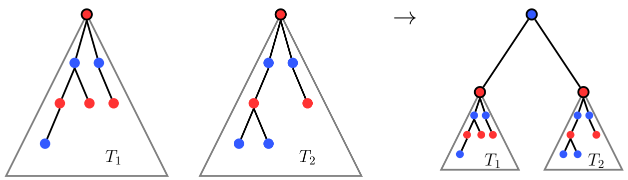
Chromatic Number $\chi(G)$**.** The minimum number of colors needed.

Lemma. Using Sequential Greedy, color$(v_i) \le \delta_i + 1$.

Theorem. Chromatic number is bounded by maximum degree.
$\chi(G) \le \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree in G, $\Delta(G) = \max_i \delta_i$.

Trees are 2-Colorable

Let us prove this for RBT's. We show that the constructor rule preserves 2-colorability.

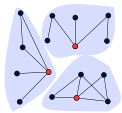


How do we know T_1's root is colored red?
How do we know T_2's root is colored red?

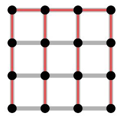
A graph is bipartite if and only if its chromatic number is 2. Trees are bipartite.

Other Graph Problems

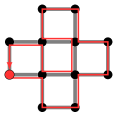
Connected Components. For "viral" marketing, pick one vertex in each *connected component* (e.g. target the "central (red)" vertices). [EASY]



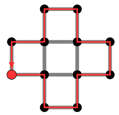
Spanning Tree. In a road grid (gray), to maintain a *minimal* "highway system" that offers high-speed travel we can use a *spanning tree* (red). [EASY]



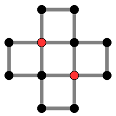
Euler Cycle. Every winter, Troy typically has a 1-foot snowfall. The snowplow should start at the depot, traverse every road *exactly once* and return to the depot, traversing an *Euler Cycle* (red). [EASY]



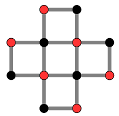
Hamiltonian Cycle. A traveling salesman starts at work and visits every house (vertex) *exactly once*, returning to work. The salesperson follows a *Hamiltonian Cycle*. [HARD]



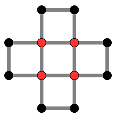
Facility Location (*K*-center). McDonalds wants to place $K = 2$ restaurants (red) in a road network so that no customer has too drive far to reach their closest McDonalds. [HARD]



Vertex Cover. Place the minimum number of policemen at intersections so that all roads can be surveiled or "covered". The policeman form a vertex cover. Can you do it with fewer than 6? [HARD]



Dominating Set. Place the fewest hospitals at intersections (vertices) so that every intersection is either at a hospital or one block away from a hospital. The red hospitals are a *dominating set*. [HARD]



Network Flow. A *source*-ISP (blue) sends packets to a *sink*-ISP (red). What is the maximum transmission rate achievable without exceeding the link capacities? We achieved flow rate 10. [EASY]