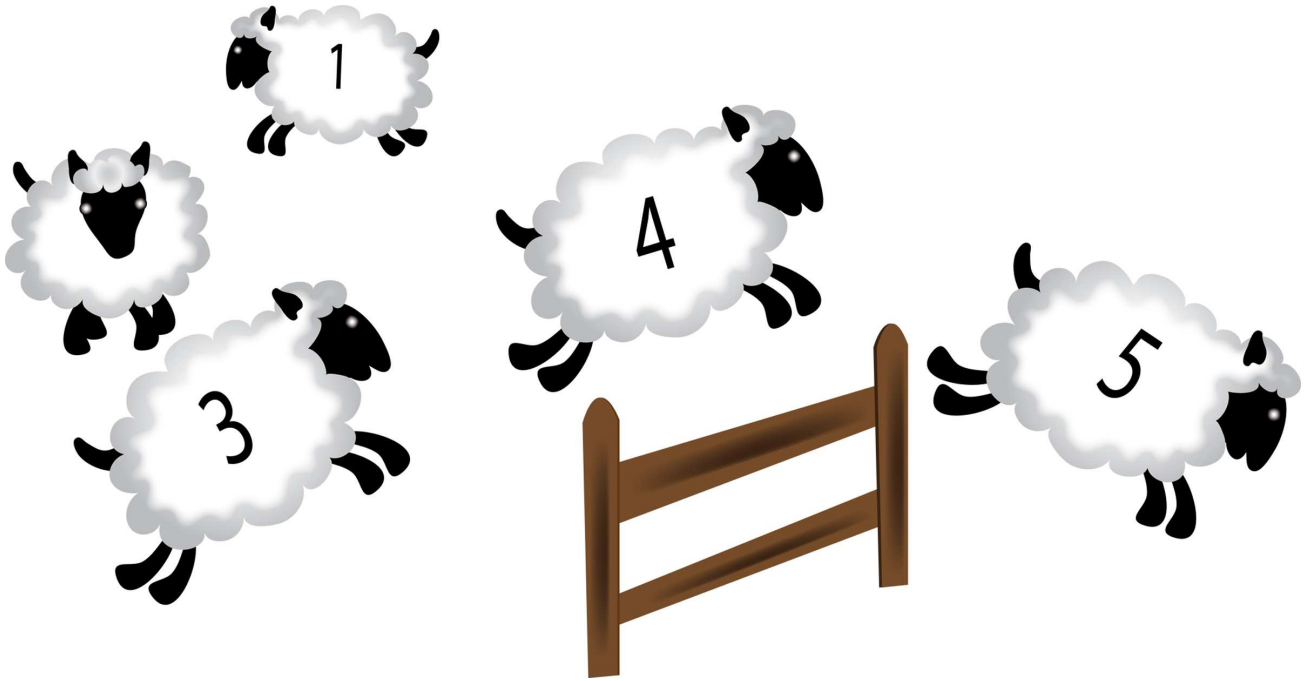
Foundations of Computer Science
Lecture 13

Counting

Counting Sequences
Build-Up Counting
Counting One Set by Counting Another: Bijection
Permutations and Combinations



Last Time

1. Be careful of what you read in the media: sex in America.
2. Bipartite Graphs and Matching (Hall's Theorem).
3. Stable Marriage.
4. Conflict Graphs and Coloring.
   - Every tree is 2-colorable.
5. Other Graph Problems
   - Connected components, spanning tree, Euler paths, network flow. ([EASY])
   - Hamiltonian paths, K-center, vertex cover, dominating set. ([HARD])

Today: Counting

1. Counting sequences.
2. Build-up counting.
3. Counting one set by counting another: bijection.
4. Permutations and combinations.

Fun Counting Fact. Radio took 38 years to reach 50M users. TV took 13 years. The World Wide Web took just 4 years.

Discrete Math Is About Objects We Can Count

Three colors of candy:

A goody-bag has 3 candies. How many distinct goody-bags?

(Only the number of each color matters: { } and {} are the "same" goody-bag.)

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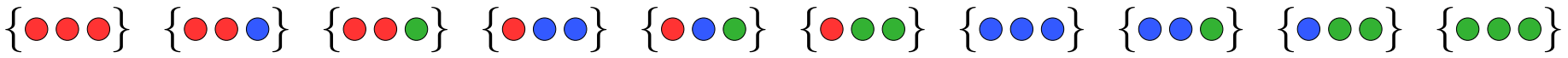

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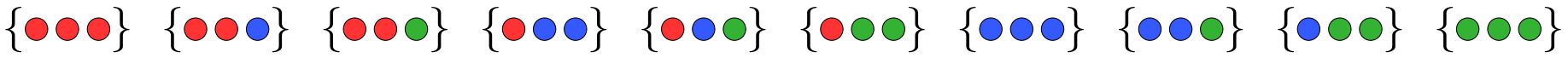

Challenge Problems.

1. What if there are 5 candies per goody-bag and 10 colors of candy?
2. Goody-bags come in bulk packs of 5. How many different bulk packs are there?

There are too many to list out. We need tools!

Sum Rule

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$$N = N_1 + N_2.$$

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Product Rule

$$|\{\, b_1 \quad b_2 \quad b_3 \,\}|$$

$$2 \times 2 \times 2$$

Let N be the number of choices for a sequence

$$x_1 x_2 x_3 \cdots x_{r-1} x_r.$$

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Example. There are 2^n binary sequences of length n: $N_1 = N_2 = \cdots = N_n = 2$.

The sum and product rules are the only basic tools we need . . . plus **TINKERING**.

Examples

1. **Menus.**
$$\begin{aligned} \text{breakfast} &\in \{\text{pancake, waffle, Doritos}\} \\ \text{lunch} &\in \{\text{burger, Doritos}\} \\ \text{dinner} &\in \{\text{salad, steak, Doritos}\} \end{aligned}$$

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1. **Menus.** breakfast $\in$ {pancake, waffle, Doritos}
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   $|\{BLD\}| = 3 \times 2 \times 3 = 18.$

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   Each student can be in or out of the committee: $2 \times 2 \times \cdots \times 2 = 2^{10} = 1024.$

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$|\{\text{committees}\}| = |\{\text{10-bit binary strings}\}|$

e.g. $\{\, s_1\ s_2\ s_3\ s_4\ s_5\ s_6\ s_7\ s_8\ s_9\ s_{10} \,\}$

1	1	0	1	0	0	0	0	1	0	$\leftrightarrow$	$\{s_1, s_2, s_4, s_9\}$
0	0	0	0	0	0	0	0	0	0	$\leftrightarrow$	$\varnothing$

Examples from CS

Challenge Exercises:

- A problem from the area of natural language processing (NLP).

  > Wikipedia has 40 million articles (6 million in English). You compute an "edit distance" between every pair of articles and store this data in a $40\text{million} \times 40\text{million}$ array of 64-bit double precision numbers.

  - Estimate how much RAM you will need to load this matrix into memory. (answer: ~ 13TB)
  - Any suggestions on feasibly performing tasks using this edit-distance data?

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- A problem in scheduling exams.

  > RPI has about 400 courses and 5000 students who induce conflicts among these courses for exam scheduling. Are 15 exam slots enough to schedule all the exams? You realize that you need to color a 400-node conflict graph usinig 15 colors.

  - You can generate and test validity of a 15-coloring at a rate of 3 million per second. Estimate how long it would take to determine that 15 exam slots is not enough. (answer: $\sim 10^{456}$years)
  - Any suggestions on what to do?

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$\binom{n}{k}$	k 0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	1						
n 3	1	3	3	1					
4									
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7									
8									

Build-Up Counting

Number of binary sequences of length n: 2^n.

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1000 1001 1010 1011 1100 1101 1110 1111

Length-5 sequences:

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01000 01001 01010 01011 01100 01101 01110 01111
10000 10001 10010 10011 10100 10101 10110 10111
11000 11001 11010 11011 11100 11101 11110 11111

$\binom{n}{k}$	k 0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4									
5									
6									
7									
8									

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4	1	4	6	4	1				
5	1	5	10	10	5	1			
6									
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5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7									
8									

(n down the left)

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base cases: $\binom{n}{0} = 1; \binom{n}{n} = 1.$

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2	1	2	1						
n 3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

Pascal's Triangle

$$\{n\text{-sequence with } k \text{ 1's}\} = 0 \bullet \underbrace{\{(n-1)\text{-sequence with } k \text{ 1's}\}}_{\binom{n-1}{k}} \quad \cup \quad 1 \bullet \underbrace{\{(n-1)\text{-sequence with } (k-1) \text{ 1's}\}}_{\binom{n-1}{k-1}}$$

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Build-up Counting for Goody Bags

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0●	1●	2●	3●	$\cdots$	n●
$Q(n,k-1)$	$Q(n-1,k-1)$	$Q(n-2,k-1)$	$Q(n-3,k-1)$	$\cdots$	$Q(0,k-1)$

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$$Q(n,k) = Q(0,k-1) + Q(1,k-1) + \cdots + Q(n,k-1) \qquad \text{(sum rule)}$$

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Build-up counting: there are $(n+1)$ types of goody-bag.

0	1	2	3	$\cdots$	n
$Q(n,k-1)$	$Q(n-1,k-1)$	$Q(n-2,k-1)$	$Q(n-3,k-1)$	$\cdots$	$Q(0,k-1)$

$$Q(n,k) = Q(0,k-1) + Q(1,k-1) + \cdots + Q(n,k-1) \qquad \text{(sum rule)}$$

$Q(n,k)$	k = 1	2	3	4	5	6	7	8	9	10	11
n = 0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
1	**1**										
2	**1**										
3	**1**										
4	**1**										
5	**1**										

$Q(n,1) = 1$; $Q(0,k) = 1$

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$Q(n,k-1)$	$Q(n-1,k-1)$	$Q(n-2,k-1)$	$Q(n-3,k-1)$	$\cdots$	$Q(0,k-1)$

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0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
1	**1**	2	3								
2	**1**	3	6								
3	**1**	4	10								
4	**1**	5	15								
5	**1**	6	21								

n

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Build-up counting: there are $(n+1)$ types of goody-bag.

0	1	2	3	$\cdots$	n
$Q(n,k-1)$	$Q(n-1,k-1)$	$Q(n-2,k-1)$	$Q(n-3,k-1)$	$\cdots$	$Q(0,k-1)$

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0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
1	**1**	2	3	4	5	6	7	8	9	10	11
2	**1**	3	6	10	15	21	28	36	45	55	66
3	**1**	4	10	20	35	56	84	120	165	220	286
4	**1**	5	15	35	70	126	210	330	495	715	1001
5	**1**	6	21	56	126	252	462	792	1287	2002	3003

(n)

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0 candies	1 candy	2 candies	3 candies	$\cdots$	n candies
$Q(n,k-1)$	$Q(n-1,k-1)$	$Q(n-2,k-1)$	$Q(n-3,k-1)$	$\cdots$	$Q(0,k-1)$

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0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
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4	**1**	5	15	35	70	126	210	330	495	715	1001
5	**1**	6	21	56	126	252	462	792	1287	2002	3003

(n = 0, 1, 2, 3, 4, 5)

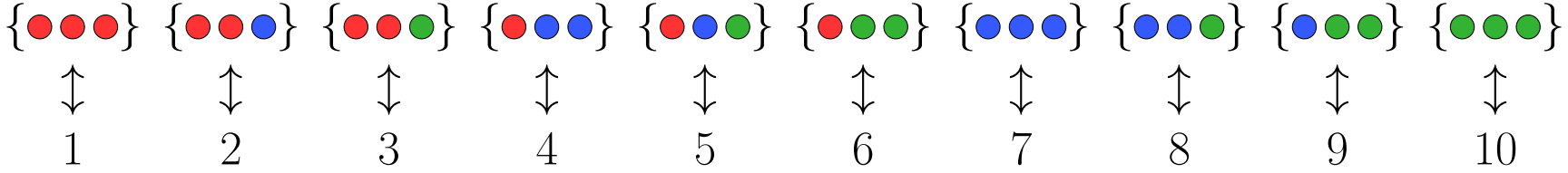
$Q(n,1) = 1$; $Q(0,k) = 1$

Challenge problems we had earlier.

1. (5 candies, 10 colors) $\to$ 2002 goody-bags.
2. How many 5 goody-bag bulk packs (goody-bags have 3 candies of 3 colors)? There are 10 types of goody-bag; 5 in a bulk pack. So we need $Q(5,10) = 2002$.

Counting One Set By Counting Another: Bijection

10 goody-bags with 3 candies of 3 colors. Can label those goody bags using $\{1, 2, \ldots, 10\}$.



1-to-1 correspondence between goody-bags and the set $\{1, 2, \ldots, 10\}$, a *bijection*.

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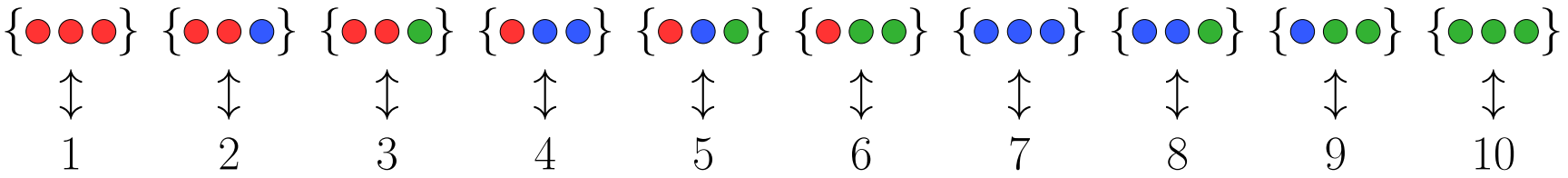

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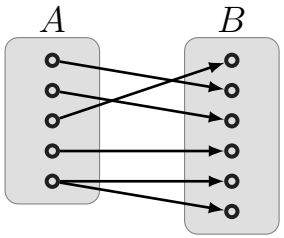
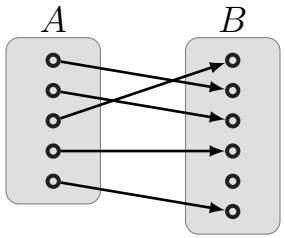
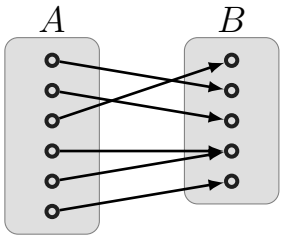


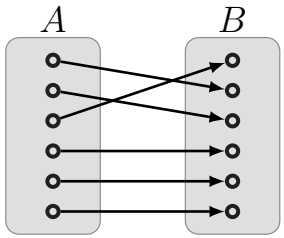
not a function



1-to-1; **not** onto
(injection, $A \overset{\text{INJ}}{\mapsto} B$)
$|A| \leq |B|$



onto; **not** 1-to-1
(surjection, $A \overset{\text{SUR}}{\mapsto} B$)
$|A| \geq |B|$



1-to-1 **and** onto
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$|A| = |B|$

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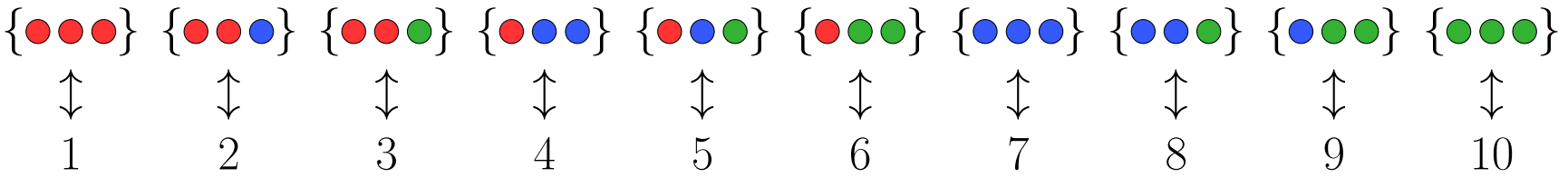

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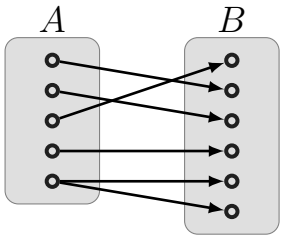

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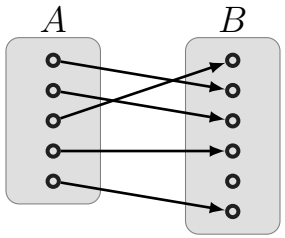

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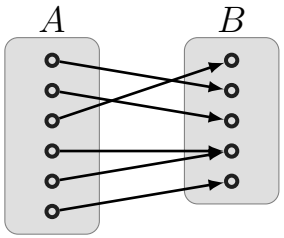

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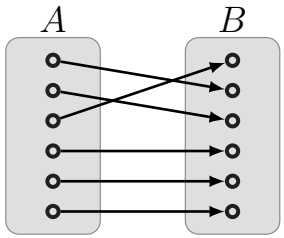


1-to-1 **and** onto
(bijection, $A \overset{\text{BIJ}}{\mapsto} B$)
$|A| = |B|$

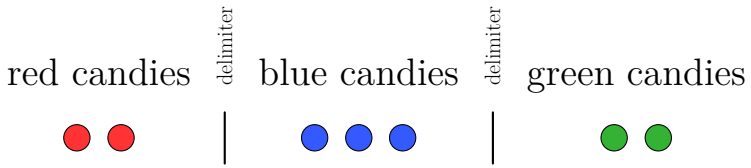
$A \overset{\text{BIJ}}{\mapsto} B$ implies $|A| = |B|$. Can count A by counting B.

Count menus by counting sequences $\{BLD\}$. Works because

- Every sequence specifies a distinct menu (1-to-1 mapping).
- Every menu corresponds to a sequence (the mapping is onto).

Goody Bags Using Bijection to Binary Sequences

3 colors, ●, ●, ●. Consider the 7-candy goody-bag $\{2●, 3●, 2●\}$:



Goody Bags Using Bijection to Binary Sequences

3 colors, ●, ●, ●. Consider the 7-candy goody-bag $\{2\bullet, 3\bullet, 2\bullet\}$:

goody-bags $\overset{\text{BIJ}}{\mapsto}$ 9-bit binary sequences with two 1's.

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Examples.

$$00100010101000 \to ○○|○○○|○|○|○○○ \leftrightarrow \{2●, 3●, 1●, 1●, 3●\}$$
$$1000011010000 \to \ |○○○○||○|○○○○ \leftrightarrow \{0●, 4●, 0●, 1●, 4●\}$$

Goody Bags Using Bijection to Binary Sequences

3 colors, 🔴, 🔵, 🟢. Consider the 7-candy goody-bag $\{2🔴, 3🔵, 2🟢\}$:

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Examples.

$$00100010101000 \rightarrow \bigcirc\bigcirc|\bigcirc\bigcirc\bigcirc|\bigcirc|\bigcirc|\bigcirc\bigcirc\bigcirc \leftrightarrow \{2🔴, 3🔵, 1🟢, 1🟣, 3🟠\}$$
$$1000011010000 \rightarrow |\bigcirc\bigcirc\bigcirc\bigcirc||\bigcirc|\bigcirc\bigcirc\bigcirc\bigcirc \leftrightarrow \{0🔴, 4🔵, 0🟢, 1🟣, 4🟠\}$$

n candies and k colors $\rightarrow (k-1)$ delimiters.

number of goody-bags with n candies of k colors = number of $(n+k-1)$-bit sequences with $(k-1)$ 1's

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Examples.

$$00100010101000 \rightarrow ○○|○○○|○|○|○○○ \leftrightarrow \{2🔴, 3🔵, 1🟢, 1🟣, 3🟠\}$$
$$1000011010000 \rightarrow \ |○○○○||○|○○○○ \leftrightarrow \{0🔴, 4🔵, 0🟢, 1🟣, 4🟠\}$$

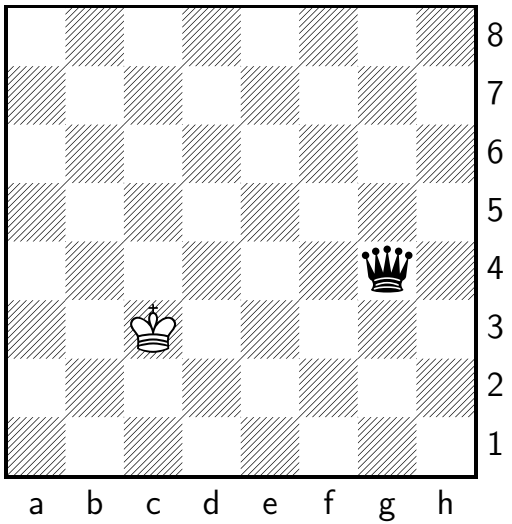
n candies and k colors $\rightarrow (k-1)$ delimiters.

number of goody-bags with n candies of k colors = number of $(n+k-1)$-bit sequences with $(k-1)$ 1's

$$Q(n,k) = \binom{n+k-1}{k-1}.$$

$\binom{n}{k}$ keeps popping up but we don't have a formula for it.

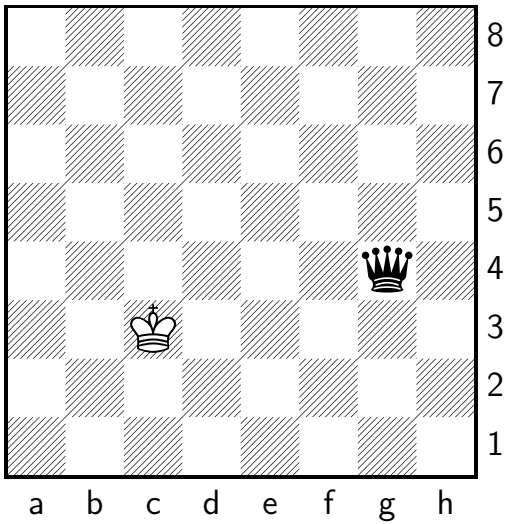
Nonaligned King-Queen Positions on a Chessboard



Nonaligned King-Queen Positions on a Chessboard

complex object	sequence to reconstruct it
king-queen position	$c3g4$
number of positions	number of sequences $c_K r_K c_Q r_Q$

Every position gives a sequence with $r_k \neq r_Q$ and $c_K \neq c_Q$
Every such sequence is a unique valid position.



Nonaligned King-Queen Positions on a Chessboard

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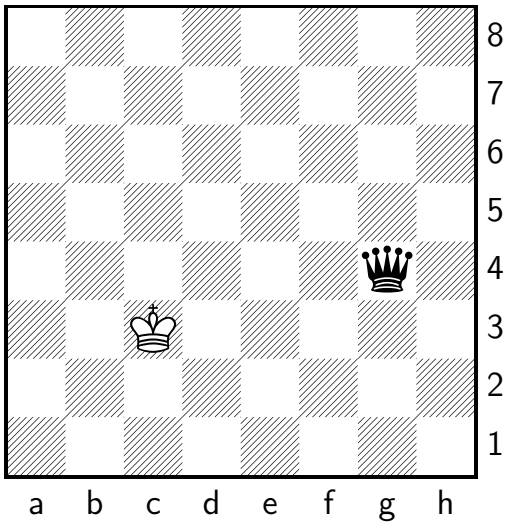

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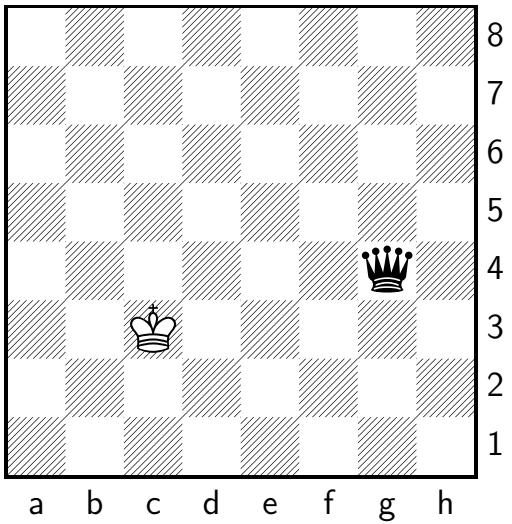

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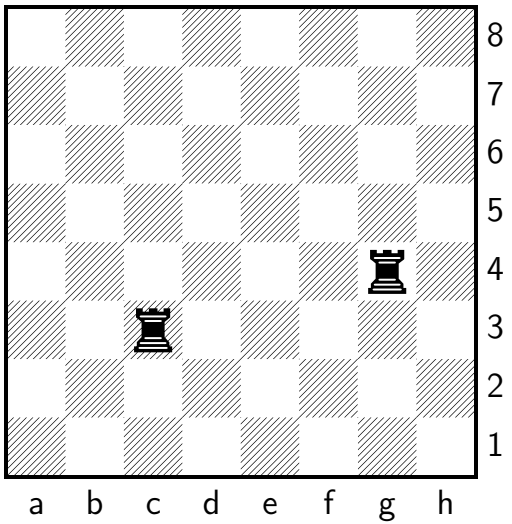
By bijection counting, number of King-Queen positions is 3136.

Nonaligned Castle-Castle Positions on a Chessboard



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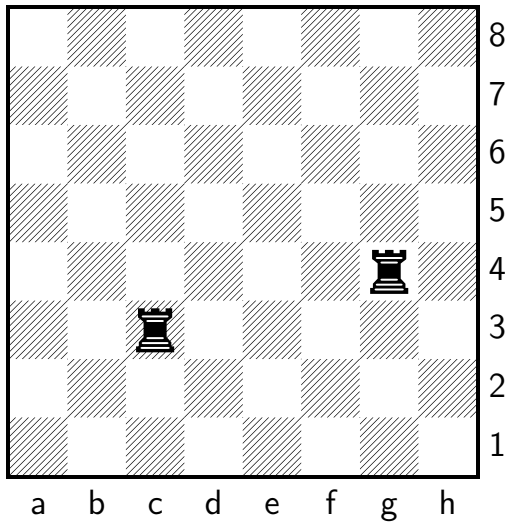

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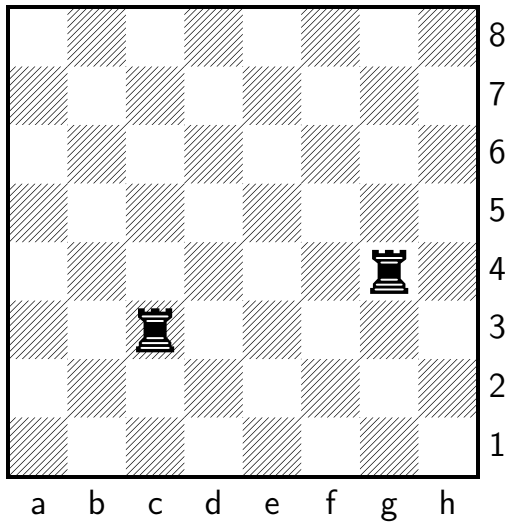

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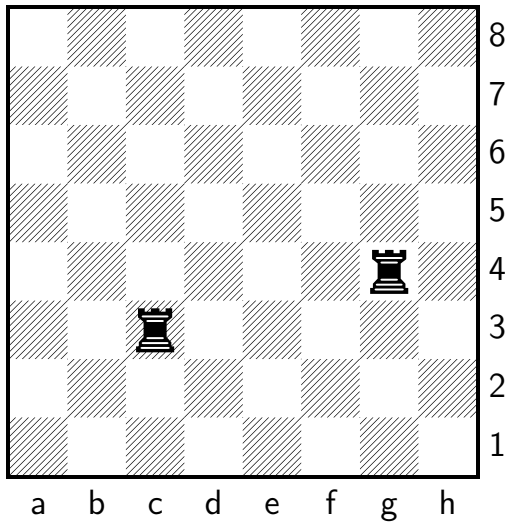

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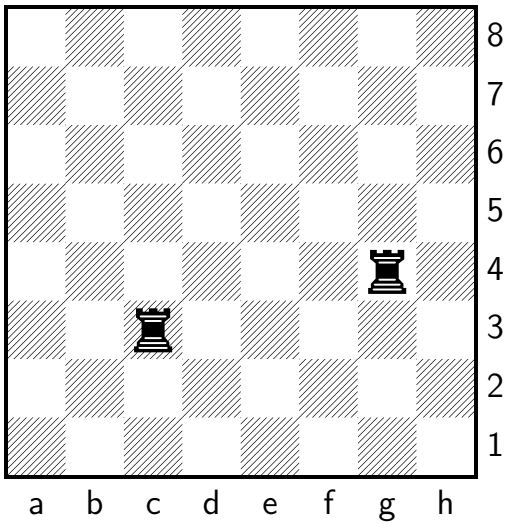

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Permutations and Combinations

$S = \{1, 2, 3, 4\}$,

the 2-orderings are:

$\{1\,2, 1\,3, 1\,4, 2\,1, 2\,3, 2\,4, 3\,1, 3\,2, 3\,4, 4\,1, 4\,2, 4\,3\}$

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Exercise. How many 10-bit binary sequences with four 1's?

Binomial Theorem: $(x+y)^n = \sum_{i=1}^{n} \binom{n}{i} x^i y^{n-i}$

$$\begin{aligned}(x+y)^3 = (x+y)(x+y)(x+y) &= xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy \\ &= x^3 + 3x^2y + 3xy^2 + y^3.\end{aligned}$$

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Example. What is the coefficient of x^7 in the expansion of $(\sqrt{x} + 2x)^{10}$

Need $(\sqrt{x})^i (2x)^{10-i} \sim x^7$, which implies $i = 6$.

The x^7 term is $\binom{10}{6} (\sqrt{x})^6 (2x)^4$

Coefficient of x^7 is $\binom{10}{6} \times 2^4 = 3360$.

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Count the number of possible *sequences* of instructions, which equals the number of complex objects.