

# Foundations of Computer Science

## Lecture 16

### Conditional Probability

Updating a Probability when New Information Arrives  
Conditional Probability Traps  
Law of Total Probability



### Last Time

- 1 Outcome-tree method for computing probability.
- 2 Probability and sets.
  - ▶ Probability space.
  - ▶ Event is a subset of outcomes.
  - ▶ Can get complex events using set (logical) operations.
- 3 Uniform probability space
  - ▶ Toss 10 coins. Each sequence (e.g. HTHHHTTTHH) has equal probability.
  - ▶ Roll 3 dice. Each sequence (e.g. (2,4,5)) has equal probability.
  - ▶ Probability of event  $\sim$  event size.
- 4 Infinite probability space.
  - ▶ Toss a coin until you get heads (possibly never ending).

### Today: Conditional Probability

- 1 New information changes a probability.
- 2 Definition of conditional probability from regular probability.
- 3 Conditional probability traps
  - Sampling bias.
  - Transposed conditional.
- 4 Law of total probability.
  - Probabilistic case-by-case analysis.

### Flu Season

- 1 Chances a random person has the flu is about 0.01 (or 1%) (*prior* probability).  
Probability of flu :  $\mathbb{P}[\text{flu}] \approx 0.01$ .
- 2 You have a slight fever – *new information*. Chances of flu “increase”.  
Probability of flu *given* fever :  $\mathbb{P}[\text{flu} \mid \text{fever}] \approx 0.4$ .
  - ▶ New information changes the prior probability to the *posterior* probability.
  - ▶ Translate posterior as “*After* you get the new information.” $\mathbb{P}[A \mid B]$  is the (updated) *conditional* probability of  $A$ , *given* the new information  $B$ .
- 3 Roommie has flu (more new information). Flu for sure, take counter-measures.  
Probability of flu *given* fever and roommie flu :  $\mathbb{P}[\text{flu} \mid \text{fever AND roommie flu}] \approx 1$ .

**Pop Quiz.** Estimate these probabilities:

$\mathbb{P}[\text{Humans alive tomorrow}]$ ,

$\mathbb{P}[\text{No Sun tomorrow}]$ ,

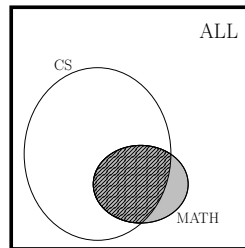
$\mathbb{P}[\text{Humans alive tomorrow} \mid \text{No Sun tomorrow}]$ .

## CS, MATH and Dual CS-MATH Majors

5,000 students: 1,000 CS; 100 MATH; 80 dual MATH-CS.

Pick a random student:

$$\begin{aligned}\mathbb{P}[\text{CS}] &= \frac{1000}{5000} = 0.2; \\ \mathbb{P}[\text{MATH}] &= \frac{100}{5000} = 0.02; \\ \mathbb{P}[\text{CS AND MATH}] &= \frac{80}{5000} = 0.016.\end{aligned}$$



New information: student is MATH. What is  $\mathbb{P}[\text{CS} | \text{MATH}]$ ?

- Effectively picking a random student from MATH.
- New probability of CS  $\sim$  striped area  $|\text{CS} \cap \text{MATH}|$ .

$$\mathbb{P}[\text{CS} | \text{MATH}] = \frac{|\text{CS} \cap \text{MATH}|}{|\text{MATH}|} = \frac{80}{100} = 0.8.$$

MATH students are 4 times more likely to be CS majors than a random student.

**Pop Quiz.** What is  $\mathbb{P}[\text{MATH} | \text{CS}]$ ? What is  $\mathbb{P}[\text{CS} | \text{CS OR MATH}]$ ? **Exercise 16.2.**

## Conditional Probability $\mathbb{P}[A | B]$

$\mathbb{P}[A | B]$  = frequency of outcomes known to be in  $B$  that are also in  $A$ .

$n_B$  outcomes in event  $B$  when you repeat an experiment  $n$  times.

$$\mathbb{P}[B] = \frac{n_B}{n}.$$

Of the  $n_B$  outcomes in  $B$ , the number also in  $A$  is  $n_{A \cap B}$ ,

$$\mathbb{P}[A \cap B] = \frac{n_{A \cap B}}{n}.$$

The frequency of outcomes in  $A$  among those outcomes in  $B$  is  $n_{A \cap B} / n_B$ ,

$$\mathbb{P}[A | B] = \frac{n_{A \cap B}}{n_B} = \frac{n_{A \cap B}}{n} \times \frac{n}{n_B} = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}.$$

$$\mathbb{P}[A | B] = \frac{n_{A \cap B}}{n_B} = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \frac{\mathbb{P}[A \text{ AND } B]}{\mathbb{P}[B]}$$

## Chances of Rain Given Clouds

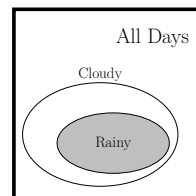
It is cloudy one in five days,  $\mathbb{P}[\text{Clouds}] = \frac{1}{5}$ . It rains one in seven days,  $\mathbb{P}[\text{Rain}] = \frac{1}{7}$ .

What are the chances of rain on a cloudy day?

$$\mathbb{P}[\text{Rain} | \text{Clouds}] = \frac{\mathbb{P}[\text{Rain} \cap \text{Clouds}]}{\mathbb{P}[\text{Clouds}]}.$$

$\{\text{Rainy Days}\} \subseteq \{\text{Cloudy Days}\} \rightarrow \mathbb{P}[\text{Rain} \cap \text{Clouds}] = \mathbb{P}[\text{Rain}]$ .

$$\mathbb{P}[\text{Rain} | \text{Clouds}] = \frac{\mathbb{P}[\text{Rain}]}{\mathbb{P}[\text{Clouds}]} = \frac{\frac{1}{7}}{\frac{1}{5}} = \frac{5}{7}.$$



5-times more likely to rain on a cloudy day than on a random day.

Crucial first step: identify the conditional probability. What is the “new information”?

## $\mathbb{P}[\text{Sum of 2 Dice is 10} | \text{Both are Odd}]$

Two dice have both rolled odd. What are the chances the sum is 10?

$$\mathbb{P}[\text{Sum is 10} | \text{Both are Odd}] = \frac{\mathbb{P}[(\text{Sum is 10}) \text{ AND } (\text{Both are Odd})]}{\mathbb{P}[\text{Both are Odd}]}$$

**Probability Space**

		1	2	3	4	5	6
Die 2 Value	6	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
5	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
4	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
3	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
2	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
1	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
Die 1 Value		1	2	3	4	5	6

$$\textcircled{1} \mathbb{P}[\text{Sum is 10}] = \frac{3}{36} = \frac{1}{12}.$$

$$\textcircled{2} \mathbb{P}[\text{Both are Odd}] = \frac{9}{36} = \frac{1}{4}.$$

$$\textcircled{3} \mathbb{P}[(\text{Sum is 10}) \text{ AND } (\text{Both are Odd})] = \frac{1}{36}.$$

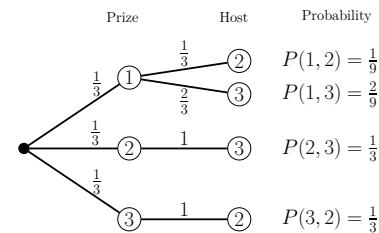
$$\textcircled{4} \mathbb{P}[\text{Sum is 10} | \text{Both are Odd}] = \frac{1}{36} \div \frac{1}{4} = \frac{1}{9}.$$

**Pop Quiz.** Compute  $\mathbb{P}[\text{Both are Odd} | \text{Sum is 10}]$ . Compare with  $\mathbb{P}[\text{Sum is 10} | \text{Both are Odd}]$ .

## Computing a Conditional Probability

- 1: Identify that you need a conditional probability  $\mathbb{P}[A | B]$ .
- 2: Determine the probability space  $(\Omega, P(\cdot))$  using the outcome-tree method.
- 3: Identify the events  $A$  and  $B$  appearing in  $\mathbb{P}[A | B]$  as subsets of  $\Omega$ .
- 4: Compute  $\mathbb{P}[A \cap B]$  and  $\mathbb{P}[B]$ .
- 5: Compute  $\mathbb{P}[A | B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$ .

## Monty Prefers Door 3



Best strategy is always switch.  
Winning outcomes: (2,3) or (3,2).

$$\mathbb{P}[\text{WinBySwitching}] = \frac{2}{3}.$$

Perk up if Monty opens door 2!

- Intuition: Why didn't Monty open door 3 if he prefers door 3?

$$\begin{aligned} \mathbb{P}[\text{Win} | \text{Monty opens Door 2}] &= \frac{\mathbb{P}[\text{Win AND Monty opens Door 2}]}{\mathbb{P}[\text{Monty opens Door 2}]} \\ &= \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{9}} \\ &= \frac{3}{4}. \end{aligned}$$

Your chances improved from  $\frac{2}{3}$  to  $\frac{3}{4}$ !

## A Pair of Boys

Your friends Ayfos, Ifar, Need and Niaz have two children each.  
What is the probability of two boys? Answer:  $\frac{1}{4}$ .

New information:

- 1 Ayfos has at least one boy. (Answer:  $\frac{1}{3}$ .)
- 2 Ifar's older child is a boy. (Answer:  $\frac{1}{2}$ .)
- 3 One day you met Need on a walk with a boy. (Answer:  $\frac{1}{2}$ .)
- 4 Niaz is Clingon. Clingons always take a son on a walk if possible. One day, you met Niaz on a walk with a boy. (Answer:  $\frac{1}{3}$ .)

Now, what is the probability of two boys in each case?

It's the same question in each case, but with slightly different additional information.  
You need conditional probabilities.

## Conditional Probability Traps

These four probabilities are all different.

$$\mathbb{P}[A] \quad \mathbb{P}[A | B] \quad \mathbb{P}[B | A] \quad \mathbb{P}[A \text{ AND } B]$$

Don't use one when you should use another.

Sampling Bias: Using  $P[A]$  instead of  $P[A | B]$

$\mathbb{P}[\text{Voter will vote Republican}] \approx \frac{1}{2}$ .  
Ask **Apple**<sup>TM</sup> to call up **i-Phone**<sup>TM</sup> users to see how they will vote.

$$\mathbb{P}[\text{Voter will vote Republican} | \text{Voter has an i-Phone}] \gg \frac{1}{2}. \quad (\text{Why?})$$

This has trapped many US election-pollers. For a famous example, **Google**<sup>TM</sup> "Dewey Defeats Truman."

Transposed Conditional: Using  $P[B | A]$  instead of  $P[A | B]$

Famous Lombard study on the riskiest profession: **Student!** Lombard confused:

$$\mathbb{P}[\text{Student} | \text{Die Young}] \quad \text{with} \quad \mathbb{P}[\text{Die Young} | \text{Student}]$$

## The LAME Test and Transposed Conditionals

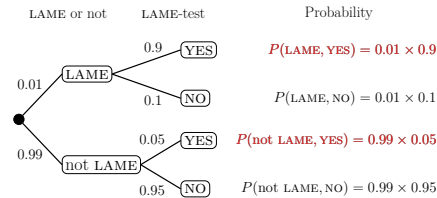
If you are LAME, the test makes a mistake in only 10% of cases.  
 If you are not LAME, the test makes a mistake in only 5% of cases.

You get tested positive. What are the chances you are LAME?

If you are not LAME, the test wouldn't make a mistake. So you are likely LAME.

It's *wrong* to look at  $\mathbb{P}[\text{positive} \mid \text{not LAME}]$ . We need  $\mathbb{P}[\text{not LAME} \mid \text{positive}]$ .

$$\begin{aligned} \mathbb{P}[\text{not LAME} \mid \text{YES}] &= \frac{\mathbb{P}[\text{not LAME AND YES}]}{\mathbb{P}[\text{YES}]} \\ &= \frac{0.99 \times 0.05}{0.99 \times 0.05 + 0.9 \times 0.01} \\ &\approx 85\%. \end{aligned}$$



The (accurate) test says YES but the chances are 85% that you are not LAME!

- You are LAME (rare) plus the test was right (likely)
- You are not LAME (very likely) plus the test got it wrong (rare). Wins!

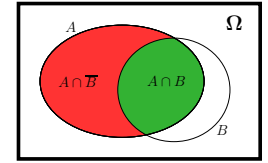
## Total Probability: Case by Case Probability

Two types of outcomes in any event  $A$ :

- The outcomes in  $B$  (green);
- The outcomes not in  $B$  (red).

$$\mathbb{P}[A] = \mathbb{P}[A \cap B] + \mathbb{P}[A \cap \bar{B}]. \quad (*)$$

(Similar to sum rule from counting.)



From the definition of conditional probability:

$$\begin{aligned} \mathbb{P}[A \cap B] &= \mathbb{P}[A \text{ AND } B] = \mathbb{P}[A \mid B] \times \mathbb{P}[B]; \\ \mathbb{P}[A \cap \bar{B}] &= \mathbb{P}[A \text{ AND } \bar{B}] = \mathbb{P}[A \mid \bar{B}] \times \mathbb{P}[\bar{B}]. \end{aligned}$$

Plugging these into (\*), we get a **FUNDAMENTAL** result for case by case analysis:

### Law of Total Probability

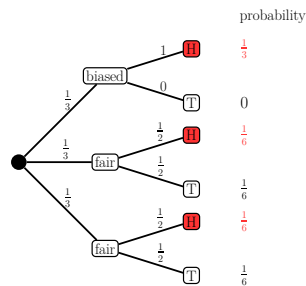
$$\mathbb{P}[A] = \mathbb{P}[A \mid B] \cdot \mathbb{P}[B] + \mathbb{P}[A \mid \bar{B}] \cdot \mathbb{P}[\bar{B}].$$

(Weight conditional probabilities for each case by probabilities of each case and add.)

## Three Coins: Two Are Fair, One is 2-Headed

Pick a random coin and flip. What is the probability of H?

### Outcome-Tree Method



$$\mathbb{P}[H] = \frac{1}{6} + \frac{1}{6} + \frac{1}{3} = \frac{2}{3}.$$

### Total Probability

Case 1.  $B$ : You picked one of the fair coins  
 Case 2.  $\bar{B}$ : You picked the two-headed coin

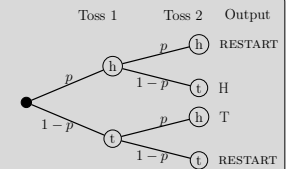
$$\begin{aligned} \mathbb{P}[H] &= \mathbb{P}[H \mid B] \cdot \mathbb{P}[B] + \mathbb{P}[H \mid \bar{B}] \cdot \mathbb{P}[\bar{B}] \\ &= \frac{1}{2} \cdot \frac{2}{3} + 1 \cdot \frac{1}{3} \\ &= \frac{2}{3}. \end{aligned}$$

**Exercise.** A box has 10 coins: 6 fair and 4 biased (probability of heads  $\frac{2}{3}$ ). What is  $\mathbb{P}[2 \text{ heads}]$  in each case?

- Pick a single random coin and flip it 3 times.
- Flip 3 times. For each flip, pick a random coin, flip it and then put the coin back.

## Fair Toss from Biased Coin (*unknown* probability $p$ of heads)?

- Make two tosses of the biased coin.  
(Lower case 'h' and 't' denote the outcomes of a toss.)
- If you get 'ht' output H; 'th' output T; otherwise RESTART.
- $P(\text{'ht'}) = P(\text{'th'}) = p(1-p)$ .
- This suggests that an H is as likely as a T.



By the law of total probability (3 cases),

$$\begin{aligned} \mathbb{P}[H] &= \mathbb{P}[H \mid \text{RESTART}] \cdot \mathbb{P}[\text{RESTART}] + \mathbb{P}[H \mid \text{'ht'}] \cdot \mathbb{P}[\text{'ht'}] + \mathbb{P}[H \mid \text{'th'}] \cdot \mathbb{P}[\text{'th'}] \\ &= \mathbb{P}[H](p^2 + (1-p)^2) + p(1-p) \end{aligned}$$

Solve for  $\mathbb{P}[H]$

$$\mathbb{P}[H] = \frac{p(1-p)}{1 - (p^2 + (1-p)^2)} = \frac{p(1-p)}{2p - 2p^2} = \frac{p(1-p)}{2p(1-p)} = \frac{1}{2}.$$

(You can also solve this problem using an infinite outcome tree and computing an infinite sum.)