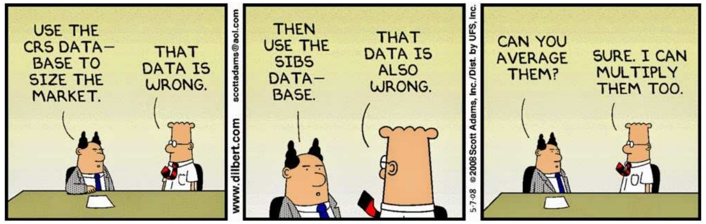
Foundations of Computer Science
Lecture 19

Expected Value

The Average Over Many Runs of an Experiment
Mathematical Expectation: A Number that Summarizes a PDF
Conditional Expectation
Law of Total Expectation



Last Time

1. Random variables.
   - PDF.
   - CDF.
   - Joint-PDF.
   - Independent random variables.
2. Important random variables.
   - Bernoulli (indicator).
   - Uniform (equalizer in strategic games).
   - Binomial (sum of Bernoullis, e.g. number of heads in n coin tosses).
   - Exponential Waiting Time Distribution (repeated tries till success).

Today: Expected Value

1. Expected value approximates the sample average.
2. Mathematical Expectation
3. Examples
   - Sum of dice.
   - Bernoulli.
   - Uniform.
   - Binomial.
   - Waiting time.
4. Conditional Expectaton
5. Law of Total Expectation

Sample Average: Toss Two Coins Many Times

	Sample Space Ω			
ω	HH	HT	TH	TT
$P(\omega)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$\mathbf{X}(\omega)$	2	1	1	0

← number of heads

→

	$x \in \mathbf{X}(\Omega)$		
x	0	1	2
$P_{\mathbf{X}}(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Toss two coins and repeat the experiment $n = 24$ times:

HH	TH	HT	HH	HH	TH	TT	TT	HH	TT	HT	HT	HH	HT	TT	HT	TT	HT	HT	TH	HH	TH	TT	TH
2	1	1	2	2	1	0	0	2	0	1	1	2	1	0	1	0	1	1	1	2	1	0	1

Average value of $\mathbf{X}$:

$$\frac{2+1+1+2+2+1+0+0+2+0+1+1+2+1+0+1+0+1+1+1+2+1+0+1}{24} = \frac{24}{24} = 1.$$

Re-order outcomes:

TT	TT	TT	TT	TT	TT	HT	HT	HT	HT	HT	HT	HT	TH	TH	TH	TH	TH	HH	HH	HH	HH	HH	HH
		$n_0 = 6$									$n_1 = 12$									$n_2 = 6$			
0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2

Average value of $\mathbf{X}$:

$$\frac{6 \times 0 + 12 \times 1 + 6 \times 2}{n} = \frac{24}{24}$$

Mathematical Expectation of a Random Variable $\mathbf{X}$

TT	TT	TT	TT	TT	TT	HT	HT	HT	HT	HT	HT	HT	TH	TH	TH	TH	TH	HH	HH	HH	HH	HH	HH
		$n_0=6$									$n_1=12$									$n_2=6$			
0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2

Average value of $\mathbf{X}$:

$$
\begin{aligned}
\frac{n_0\times 0+n_1\times 1+n_2\times 2+n_3\times 3}{n} &= \underset{\approx P_{\mathbf{X}}(0)}{\underset{\uparrow}{\frac{n_0}{n}}}\times 0+\underset{\approx P_{\mathbf{X}}(1)}{\underset{\uparrow}{\frac{n_1}{n}}}\times 1+\underset{\approx P_{\mathbf{X}}(2)}{\underset{\uparrow}{\frac{n_2}{n}}}\times 2 \\
&\approx P_{\mathbf{X}}(0)\times 0+P_{\mathbf{X}}(1)\times 1+P_{\mathbf{X}}(2)\times 2 \\
&= \sum_{x\in\mathbf{X}(\Omega)} x\cdot P_{\mathbf{X}}(x)
\end{aligned}
$$

For two coins the expected value is $0\times\frac{1}{4}+1\times\frac{1}{2}+2\times\frac{1}{4}=1$.

Add the possible values x weighted by their probabilities $P_{\mathbf{X}}(x)$,

$$\mathbb{E}[\mathbf{X}]=\sum_{x\in\mathbf{X}(\Omega)} x\cdot P_{\mathbf{X}}(x).$$

Synonyms: Expectation; Expected Value; Mean; Average.
Important Exercise. Show that $\mathbb{E}[\mathbf{X}]=\sum_{\omega\in\Omega}\mathbf{X}(\omega)P(\omega)$.

Expected Number of Heads from 3 Coin Tosses is $1\frac{1}{2}$!

$$\mathbb{E}[\mathbf{X}]=\sum_{x\in\mathbf{X}(\Omega)} x\cdot P_{\mathbf{X}}(x).$$

	Sample Space Ω							
ω	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
$P(\omega)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$\mathbf{X}(\omega)$	3	2	2	1	2	1	1	0

→

	$x\in\mathbf{X}(\Omega)$			
x	0	1	2	3
$P_{\mathbf{X}}(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$
\begin{aligned}
\mathbb{E}[\text{number heads}] &= 0\times\tfrac{1}{8}+1\times\tfrac{3}{8}+2\times\tfrac{3}{8}+3\times 18 \\
&= \tfrac{12}{8} \\
&= 1\tfrac{1}{2}.
\end{aligned}
$$

What does this mean?!?

Exercise. Let $\mathbf{X}$ be the the value of a fair die roll. Show that $\mathbb{E}[\mathbf{X}]=3\frac{1}{2}$.

Expected Sum of Two Dice

Probability Space

Die 2 Value						
⚅	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
⚄	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
⚃	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
⚂	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
⚁	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
⚀	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
Die 1 Value	⚀	⚁	⚂	⚃	⚄	⚅

$\mathbf{X}$ = sum

⚅	7	8	9	10	11	12
⚄	6	7	8	9	10	11
⚃	5	6	7	8	9	10
⚂	4	5	6	7	8	9
⚁	3	4	5	6	7	8
⚀	2	3	4	5	6	7
Die 1 Value	⚀	⚁	⚂	⚃	⚄	⚅

x	2	3	4	5	6	7	8	9	10	11	12
$P_{\mathbf{X}}(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{1}{36}$

$$
\begin{aligned}
\mathbb{E}[\mathbf{X}] &= \sum_x x\cdot P_{\mathbf{X}}(x) \\
&= \tfrac{1}{36}(2\times 1+3\times 2+4\times 3+5\times 4+6\times 5+7\times 6+8\times 5+9\times 4+10\times 3+11\times 2+12\times 1) \\
&= \tfrac{252}{36}=7.
\end{aligned}
$$

(Expected sum of two dice is twice the expected roll of one die.)

Expected Value of a Bernoulli Random Variable

A Bernoulli random variable $\mathbf{X}$ takes a value in $\{0,1\}$

x	0	1
$P_{\mathbf{X}}(x)$	$1-p$	p

The expected value is

$$\mathbb{E}[\mathbf{X}]=0\cdot(1-p)+1\cdot p=p.$$

A Bernoulli random variable with success probability p has expected value p.

Expected Value of a Uniform Random Variable

A uniform random variable $\mathbf{X}$ takes values in $\{1, \ldots, n\}$,

x	1	2	3	4	$\cdots$	$n-1$	n
$P_{\mathbf{X}}(x)$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$

The expected value is

$$\begin{aligned}\mathbb{E}[\mathbf{X}] &= \tfrac{1}{n} + \tfrac{2}{n} + \tfrac{3}{n} + \cdots + \tfrac{n}{n} \\ &= \tfrac{1}{n}(1 + 2 + \cdots + n) \\ &= \tfrac{1}{n} \times \tfrac{1}{2}n(n+1).\end{aligned}$$

A uniform random variable on $[1, n]$ has expected value $= \frac{1}{2}(n+1)$.

What is the Expected Number of Heads in n Coin Tosses?

Binomial distribution: $P_{\mathbf{X}}(k) = B(k; n, p) = \binom{n}{k} p^k (1-p)^{n-k}$.

x	0	1	2	$\ldots$	k	$\ldots$	n
$P_{\mathbf{X}}(x)$	$\binom{n}{0} p^0 (1-p)^n$	$\binom{n}{1} p^1 (1-p)^{n-1}$	$\binom{n}{2} p^2 (1-p)^{n-2}$	$\ldots$	$\binom{n}{k} p^k (1-p)^{n-k}$	$\ldots$	$\binom{n}{n} p^n (1-p)^0$

$$\mathbb{E}[\mathbf{X}] = 0 \cdot \binom{n}{0} p^0 q^n + 1 \cdot \binom{n}{1} p^1 q^{n-1} + \cdots + k \cdot \binom{n}{k} p^k q^{n-k} + \cdots + n \cdot \binom{n}{n} p^n q^0 \qquad (q = 1-p)$$

Binomial Theorem:

$$\begin{aligned}(p+q)^n &= \binom{n}{0} p^0 q^n + \binom{n}{1} p^1 q^{n-1} + \cdots + \binom{n}{k} p^k q^{n-k} + \cdots + \binom{n}{n} p^n q^0 \\ \xrightarrow{\frac{d}{dp}} \quad n(p+q)^{n-1} &= 1 \cdot \binom{n}{1} p^0 q^{n-1} + 2 \cdot \binom{n}{2} p^1 q^{n-2} + \cdots + k \cdot \binom{n}{k} p^{k-1} q^{n-k} + \cdots + n \cdot \binom{n}{n} p^{n-1} q^0 \\ \xrightarrow{\times p} \quad np\underbrace{(p+q)^{n-1}}_{1} &= 1 \cdot \binom{n}{1} p^1 q^{n-1} + 2 \cdot \binom{n}{2} p^2 q^{n-2} + \cdots + k \cdot \binom{n}{k} p^k q^{n-k} + \cdots + n \cdot \binom{n}{n} p^n q^0\end{aligned}$$

Expected number of heads in n biased coin tosses is np.

Example. Answer randomly 15 multiple choice questions with 5 choices ($p = \frac{1}{5}$): expect to get $15 \times \frac{1}{5} = 3$ correct.

Expected Waiting Time to Success

Exponential Waiting Time Distribution: $P_{\mathbf{X}}(t) = \beta(1-p)^t$.

t	1	2	3	$\ldots$	k	$\ldots$
$P_{\mathbf{X}}(t)$	$\beta(1-p)$	$\beta(1-p)^2$	$\beta(1-p)^3$	$\ldots$	$\beta(1-p)^k$	$\ldots$

$(\beta = p/(1-p))$

$$\mathbb{E}[\mathbf{X}] = \beta(1 \cdot (1-p)^1 + 2 \cdot (1-p)^2 + 3 \cdot (1-p)^3 + \cdots + k \cdot (1-p)^k + \cdots)$$

Geometric series formula:

$$\begin{aligned}\frac{1}{1-a} &= 1 + a + a^2 + a^3 + a^4 + \cdots \\ \xrightarrow{\frac{d}{da}} \quad \frac{1}{(1-a)^2} &= 1 + 2 \cdot a + 3 \cdot a^2 + 4 \cdot a^3 + 5a \cdot^4 + \cdots \\ \xrightarrow{\times a} \quad \frac{a}{(1-a)^2} &= 1 \cdot a^1 + 2 \cdot a^2 + 3 \cdot a^3 + 4 \cdot a^4 + 5a \cdot^5 + \cdots \qquad (a = 1-p) \\ \rightarrow \quad \mathbb{E}[\mathbf{X}] &= \beta \times \frac{1-p}{p^2} = \frac{1}{p}.\end{aligned}$$

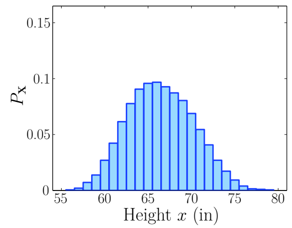
Expected waiting time is $1/p$.

Exercise. A couple who is waiting for a boy expects to make 2 trials (children).

Conditional Expectation: Expected Height of Men

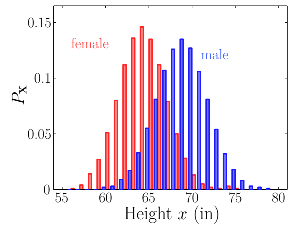
New information changes a probability. Hence, the expected value also changes.

Height Distribution



$\mathbb{E}[\text{height}] \approx 66\frac{1}{2}$".

Conditional Height Distribution



$\mathbb{E}[\text{height} \mid \text{female}] \approx 64$" (red)

$\mathbb{E}[\text{height} \mid \text{male}] \approx 69$" (blue)

Conditional Expected Value $\mathbb{E}[\mathbf{X} \mid A]$:

$$\mathbb{E}[\mathbf{X} \mid A] = \sum_{x \in \mathbf{X}(\Omega)} x \cdot \mathbb{P}[\mathbf{X} = x \mid A] = \sum_{x \in \mathbf{X}(\Omega)} x \cdot P_{\mathbf{X}}(x \mid A).$$

Law of Total Expectation

Case by case analysis for expectation (similar to the Law of Total Probability).

Law of Total Expectation

$$\mathbb{E}[\mathbf{X}] = \mathbb{E}[\mathbf{X} \mid A] \cdot \mathbb{P}[A] + \mathbb{E}[\mathbf{X} \mid \overline{A}] \cdot \mathbb{P}[\overline{A}].$$

$$\begin{aligned}
\mathbb{E}[\text{height}] &= \underset{\substack{\uparrow \\ 69''}}{\mathbb{E}[\text{height} \mid \text{male}]}\ \underset{\substack{\uparrow \\ 0.49}}{\mathbb{P}[\text{male}]} + \underset{\substack{\uparrow \\ 64''}}{\mathbb{E}[\text{height} \mid \text{female}]}\ \underset{\substack{\uparrow \\ 0.51}}{\mathbb{P}[\text{female}]} \\
&= 69 \times 0.49 + 64 \times 0.51 \\
&\approx 66\tfrac{1}{2}''.
\end{aligned}$$

Example

A jar has 9 fair coins and 1 two-headed coin.

Choose a random coin and flip it 10 times.

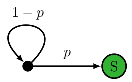
$\mathbf{X}$ is the number of heads.

$$\begin{aligned}
\mathbb{E}[\mathbf{X}] &= \underset{\substack{\uparrow \\ 10 \times \frac{1}{2}}}{\mathbb{E}[\mathbf{X} \mid \text{fair}]}\ \underset{\substack{\uparrow \\ \frac{9}{10}}}{\mathbb{P}[\text{fair}]} + \underset{\substack{\uparrow \\ 10}}{\mathbb{E}[\mathbf{X} \mid \text{2-headed}]}\ \underset{\substack{\uparrow \\ \frac{1}{10}}}{\mathbb{P}[\text{2-headed}]} \\
&= 5 \times \frac{9}{10} + 10 \times \frac{1}{10} \\
&= 5\frac{1}{2}.
\end{aligned}$$

Expected Waiting Time from Law of Total Expectation

$\mathbf{X}$ is the waiting time. Two cases:

- First trial is a succeeds (S) with probability p, i.e., $\mathbf{X} = 1$.
- First trial is a fails (F) with probability $1 - p$, i.e., "restart".



$$\begin{aligned}
\mathbb{E}[\mathbf{X}] &= \underset{\substack{\uparrow \\ 1}}{\mathbb{E}[\mathbf{X} \mid \text{S}]}\ \underset{\substack{\uparrow \\ p}}{\mathbb{P}[\text{S}]} + \underset{\substack{\uparrow \\ 1+\mathbb{E}[\mathbf{X}]}}{\mathbb{E}[\mathbf{X} \mid \text{F}]}\ \underset{\substack{\uparrow \\ (1-p)}}{\mathbb{P}[\text{F}]} \\
&= p \cdot 1 + (1-p) \cdot (1 + \mathbb{E}[\mathbf{X}]) \\
&= 1 + (1-p) \cdot \mathbb{E}[\mathbf{X}].
\end{aligned}$$

Solve for $\mathbb{E}[\mathbf{X}]$,

$$\mathbb{E}[\mathbf{X}] = \frac{1}{p}.$$

Practice. Exercise 6.5 and 6.8.