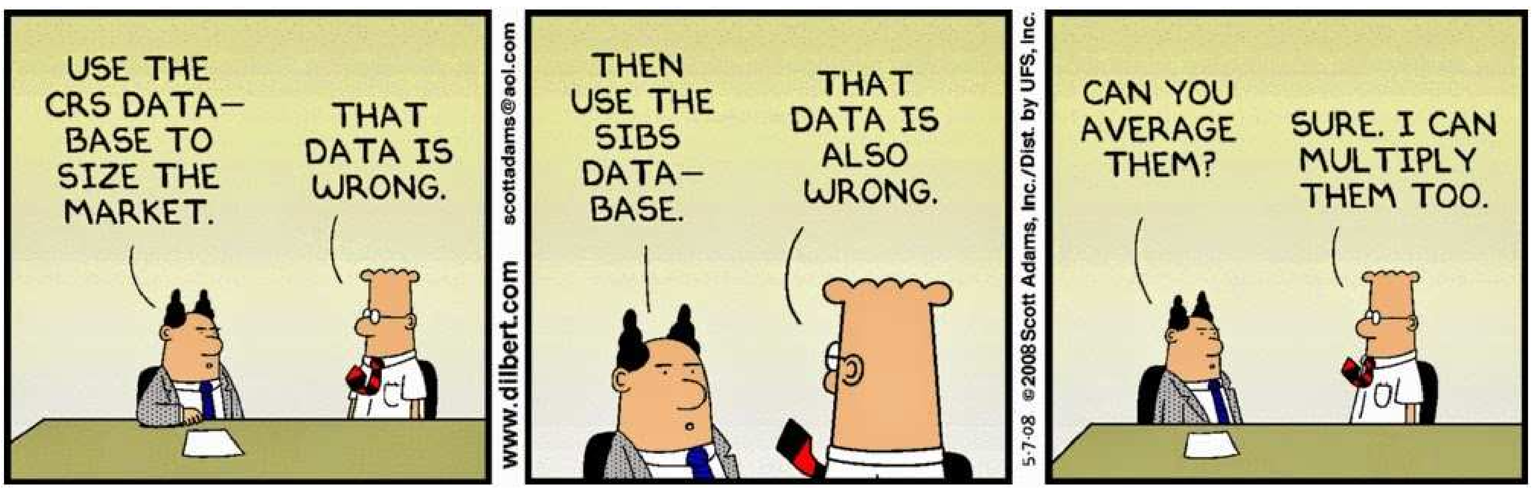
Foundations of Computer Science
Lecture 19

Expected Value

The Average Over Many Runs of an Experiment
Mathematical Expectation: A Number that Summarizes a PDF
Conditional Expectation
Law of Total Expectation



Last Time

1. Random variables.
   - PDF.
   - CDF.
   - Joint-PDF.
   - Independent random variables.

2. Important random variables.
   - Bernoulli (indicator).
   - Uniform (equalizer in strategic games).
   - Binomial (sum of Bernoullis, e.g. number of heads in n coin tosses).
   - Exponential Waiting Time Distribution (repeated tries till success).

Today: Expected Value

1. Expected value approximates the sample average.
2. Mathematical Expectation
3. Examples
   - Sum of dice.
   - Bernoulli.
   - Uniform.
   - Binomial.
   - Waiting time.
4. Conditional Expectaton
5. Law of Total Expectation

Sample Average: Toss Two Coins Many Times

	Sample Space Ω			
ω	HH	HT	TH	TT
$P(\omega)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$\mathbf{X}(\omega)$	2	1	1	0

$\leftarrow$ number of heads

$\rightarrow$

	$x \in \mathbf{X}(\Omega)$		
x	0	1	2
$P_{\mathbf{X}}(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Toss two coins and repeat the experiment $n = 24$ times:

HH	TH	HT	HH	HH	TH	TT	TT	HH	TT	HT	HT	HH	HT	TT	HT	TT	HT	HT	TH	HH	TH	TT	TH
2	1	1	2	2	1	0	0	2	0	1	1	2	1	0	1	0	1	1	1	2	1	0	1

Average value of $\mathbf{X}$:

$$\frac{2+1+1+2+2+1+0+0+2+0+1+1+2+1+0+1+0+1+1+1+2+1+0+1}{24} = \frac{24}{24} = 1.$$

Re-order outcomes:

TT	TT	TT	TT	TT	TT	HT	HT	HT	HT	HT	HT	HT	TH	TH	TH	TH	TH	HH	HH	HH	HH	HH	HH
		$n_0 = 6$										$n_1 = 12$								$n_2 = 6$			
0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2

Average value of $\mathbf{X}$:

$$\frac{6 \times 0 + 12 \times 1 + 6 \times 2}{n} = \frac{24}{24}$$

Mathematical Expectation of a Random Variable $\mathbf{X}$

TT	TT	TT	TT	TT	TT	HT	HT	HT	HT	HT	HT	HT	TH	TH	TH	TH	TH	HH	HH	HH	HH	HH	HH
		$n_0 = 6$									$n_1 = 12$										$n_2 = 6$		
0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2

Average value of $\mathbf{X}$:

$$\begin{aligned}\frac{n_0 \times 0 + n_1 \times 1 + n_2 \times 2 + n_3 \times 3}{n} &= \underset{\approx P_{\mathbf{X}}(0)}{\underset{\uparrow}{\frac{n_0}{n}}} \times 0 + \underset{\approx P_{\mathbf{X}}(1)}{\underset{\uparrow}{\frac{n_1}{n}}} \times 1 + \underset{\approx P_{\mathbf{X}}(2)}{\underset{\uparrow}{\frac{n_2}{n}}} \times 2 \\ &\approx P_{\mathbf{X}}(0) \times 0 + P_{\mathbf{X}}(1) \times 1 + P_{\mathbf{X}}(2) \times 2 \\ &= \sum_{x \in \mathbf{X}(\Omega)} x \cdot P_{\mathbf{X}}(x)\end{aligned}$$

For two coins the expected value is $0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} = 1$.

Add the possible values x weighted by their probabilities $P_{\mathbf{X}}(x)$,

$$\mathbb{E}[\mathbf{X}] = \sum_{x \in \mathbf{X}(\Omega)} x \cdot P_{\mathbf{X}}(x).$$

Synonyms: Expectation; Expected Value; Mean; Average.

Important Exercise. Show that $\mathbb{E}[\mathbf{X}] = \sum_{\omega \in \Omega} \mathbf{X}(\omega) P(\omega)$.

Expected Number of Heads from 3 Coin Tosses is $1\frac{1}{2}$!

$$\mathbb{E}[\mathbf{X}] = \sum_{x \in \mathbf{X}(\Omega)} x \cdot P_{\mathbf{X}}(x).$$

	Sample Space Ω							
ω	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
$P(\omega)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$\mathbf{X}(\omega)$	3	2	2	1	2	1	1	0

$\rightarrow$

	$x \in \mathbf{X}(\Omega)$			
x	0	1	2	3
$P_{\mathbf{X}}(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$\begin{aligned}\mathbb{E}[\text{number heads}] &= 0 \times \tfrac{1}{8} + 1 \times \tfrac{3}{8} + 2 \times \tfrac{3}{8} + 3 \times 18 \\ &= \tfrac{12}{8} \\ &= 1\tfrac{1}{2}.\end{aligned}$$

What does this mean?!?

Exercise. Let $\mathbf{X}$ be the the value of a fair die roll. Show that $\mathbb{E}[\mathbf{X}] = 3\frac{1}{2}$.

Expected Sum of Two Dice

Probability Space

Die 2 Value						
⚅	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
⚄	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
⚃	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
⚂	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
⚁	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
⚀	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
	⚀	⚁	⚂	⚃	⚄	⚅

Die 1 Value

X = sum

⚅	7	8	9	10	11	12
⚄	6	7	8	9	10	11
⚃	5	6	7	8	9	10
⚂	4	5	6	7	8	9
⚁	3	4	5	6	7	8
⚀	2	3	4	5	6	7
	⚀	⚁	⚂	⚃	⚄	⚅

Die 1 Value

x	2	3	4	5	6	7	8	9	10	11	12
$P_{\mathbf{X}}(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{1}{36}$

$$\begin{aligned}\mathbb{E}[\mathbf{X}] &= \sum_x x \cdot P_{\mathbf{X}}(x) \\ &= \tfrac{1}{36}(2\times1+3\times2+4\times3+5\times4+6\times5+7\times6+8\times5+9\times4+10\times3+11\times2+12\times1) \\ &= \tfrac{252}{36} = 7.\end{aligned}$$

(Expected sum of two dice is twice the expected roll of one die.)

Expected Value of a Bernoulli Random Variable

A Bernoulli random variable $\mathbf{X}$ takes a value in $\{0, 1\}$

$$\begin{array}{r|cc} x & 0 & 1 \\ P_{\mathbf{X}}(x) & 1-p & p \end{array}$$

The expected value is

$$\mathbb{E}[\mathbf{X}] = 0 \cdot (1-p) + 1 \cdot p = p.$$

A Bernoulli random variable with success probability p has expected value p.

Expected Value of a Uniform Random Variable

A uniform random variable $\mathbf{X}$ takes values in $\{1, \ldots, n\}$,

x	1	2	3	4	$\cdots$	$n-1$	n
$P_{\mathbf{X}}(x)$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$

The expected value is

$$\begin{aligned}\mathbb{E}[\mathbf{X}] &= \tfrac{1}{n} + \tfrac{2}{n} + \tfrac{3}{n} + \cdots + \tfrac{n}{n} \\ &= \tfrac{1}{n}(1 + 2 + \cdots + n) \\ &= \tfrac{1}{n} \times \tfrac{1}{2} n(n+1).\end{aligned}$$

A uniform random variable on $[1, n]$ has expected value $= \frac{1}{2}(n+1)$.

What is the Expected Number of Heads in n Coin Tosses?

Binomial distribution: $P_{\mathbf{X}}(k) = B(k; n, p) = \binom{n}{k} p^k (1-p)^{n-k}$.

x	0	1	2	…	k	…	n
$P_{\mathbf{X}}(x)$	$\binom{n}{0} p^0(1-p)^n$	$\binom{n}{1} p^1(1-p)^{n-1}$	$\binom{n}{2} p^2(1-p)^{n-2}$	…	$\binom{n}{k} p^k(1-p)^{n-k}$	…	$\binom{n}{n} p^n(1-p)^0$

$$\mathbb{E}[\mathbf{X}] = 0\cdot\binom{n}{0} p^0 q^n + 1\cdot\binom{n}{1} p^1 q^{n-1} + \cdots + k\cdot\binom{n}{k} p^k q^{n-k} + \cdots + n\cdot\binom{n}{n} p^n q^0 \qquad (q = 1-p)$$

Binomial Theorem:

$$
\begin{aligned}
(p+q)^n &= \binom{n}{0} p^0 q^n + \binom{n}{1} p^1 q^{n-1} + \cdots + \binom{n}{k} p^k q^{n-k} + \cdots + \binom{n}{n} p^n q^0 \\
\xrightarrow{\frac{d}{dp}} \quad n(p+q)^{n-1} &= 1\cdot\binom{n}{1} p^0 q^{n-1} + 2\cdot\binom{n}{2} p^1 q^{n-2} + \cdots + k\cdot\binom{n}{k} p^{k-1} q^{n-k} + \cdots + n\cdot\binom{n}{n} p^{n-1} q^0 \\
\xrightarrow{\times p} \quad np\,\underbrace{(p+q)^{n-1}}_{1} &= 1\cdot\binom{n}{1} p^1 q^{n-1} + 2\cdot\binom{n}{2} p^2 q^{n-2} + \cdots + k\cdot\binom{n}{k} p^k q^{n-k} + \cdots + n\cdot\binom{n}{n} p^n q^0
\end{aligned}
$$

Expected number of heads in n biased coin tosses is np.

Example. Answer randomly 15 multiple choice questions with 5 choices ($p = \frac{1}{5}$): expect to get $15 \times \frac{1}{5} = 3$ correct.

Expected Waiting Time to Success

Exponential Waiting Time Distribution: $P_{\mathbf{X}}(t) = \beta(1-p)^t$.

t	1	2	3	$\ldots$	k	$\ldots$
$P_{\mathbf{X}}(t)$	$\beta(1-p)$	$\beta(1-p)^2$	$\beta(1-p)^3$	$\ldots$	$\beta(1-p)^k$	$\ldots$

$(\beta = p/(1-p))$

$$\mathbb{E}[\mathbf{X}] = \beta(1\cdot(1-p)^1 + 2\cdot(1-p)^2 + 3\cdot(1-p)^3 + \cdots + k\cdot(1-p)^k + \cdots)$$

Geometric series formula:

$$\begin{aligned}
& \frac{1}{1-a} &=& \; 1 + a + a^2 + a^3 + a^4 + \cdots \\
\xrightarrow{\frac{d}{da}} \; & \frac{1}{(1-a)^2} &=& \; 1 + 2\cdot a + 3\cdot a^2 + 4\cdot a^3 + 5a\cdot^4 + \cdots \\
\xrightarrow{\times a} \; & \frac{a}{(1-a)^2} &=& \; 1\cdot a^1 + 2\cdot a^2 + 3\cdot a^3 + 4\cdot a^4 + 5a\cdot^5 + \cdots \qquad (a = 1-p) \\
\rightarrow \; & \mathbb{E}[\mathbf{X}] &=& \; \beta \times \frac{1-p}{p^2} = \frac{1}{p}.
\end{aligned}$$

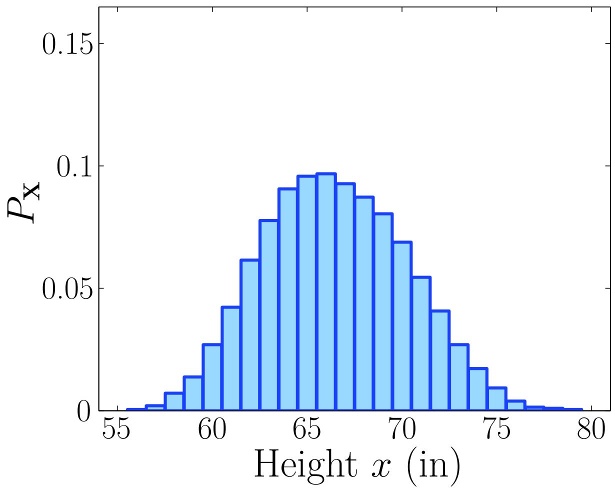
Expected waiting time is $1/p$.

Exercise. A couple who is waiting for a boy expects to make 2 trials (children).

Conditional Expectation: Expected Height of Men

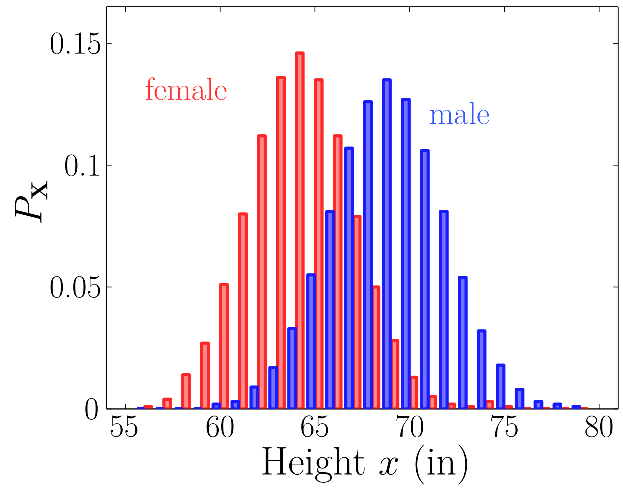
New information changes a probability. Hence, the expected value also changes.

Height Distribution



$\mathbb{E}[\text{height}] \approx 66\frac{1}{2}''$.

Conditional Height Distribution



$$\begin{aligned}\mathbb{E}[\text{height} \mid \text{female}] &\approx 64'' \text{ (red)}\\ \mathbb{E}[\text{height} \mid \text{male}] &\approx 69'' \text{ (blue)}\end{aligned}$$

Conditional Expected Value $\mathbb{E}[\mathbf{X} \mid A]$:

$$\mathbb{E}[\mathbf{X} \mid A] = \sum_{x \in \mathbf{X}(\Omega)} x \cdot \mathbb{P}[\mathbf{X} = x \mid A] = \sum_{x \in \mathbf{X}(\Omega)} x \cdot P_\mathbf{X}(x \mid A).$$

Law of Total Expectation

Case by case analysis for expectation (similar to the Law of Total Probability).

Law of Total Expectation

$$\mathbb{E}[\mathbf{X}] = \mathbb{E}[\mathbf{X} \mid A] \cdot \mathbb{P}[A] + \mathbb{E}[\mathbf{X} \mid \overline{A}] \cdot \mathbb{P}[\overline{A}].$$

$$\begin{aligned}
\mathbb{E}[\text{height}] &= \underset{\substack{\uparrow \\ 69''}}{\mathbb{E}[\text{height} \mid \text{male}]}\ \underset{\substack{\uparrow \\ 0.49}}{\mathbb{P}[\text{male}]} + \underset{\substack{\uparrow \\ 64''}}{\mathbb{E}[\text{height} \mid \text{female}]}\ \underset{\substack{\uparrow \\ 0.51}}{\mathbb{P}[\text{female}]} \\
&= 69 \times 0.49 + 64 \times 0.51 \\
&\approx 66\tfrac{1}{2}''.
\end{aligned}$$

Example

A jar has 9 fair coins and 1 two-headed coin.

Choose a random coin and flip it 10 times.

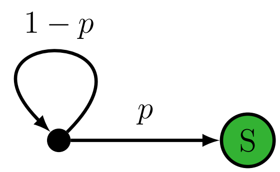
X is the number of heads.

$$
\begin{aligned}
\mathbb{E}[\mathbf{X}] &= \underset{\substack{\uparrow \\ 10\times\frac{1}{2}}}{\mathbb{E}[\mathbf{X} \mid \text{fair}]}\ \underset{\substack{\uparrow \\ \frac{9}{10}}}{\mathbb{P}[\text{fair}]} + \underset{\substack{\uparrow \\ 10}}{\mathbb{E}[\mathbf{X} \mid \text{2-headed}]}\ \underset{\substack{\uparrow \\ \frac{1}{10}}}{\mathbb{P}[\text{2-headed}]} \\
&= 5 \times \frac{9}{10} + 10 \times \frac{1}{10} \\
&= 5\frac{1}{2}.
\end{aligned}
$$

Expected Waiting Time from Law of Total Expectation

$\mathbf{X}$ is the waiting time. Two cases:

- First trial is a succeeds (S) with probability p, i.e., $\mathbf{X} = 1$.
- First trial is a fails (F) with probability $1 - p$, i.e., "restart".



$$
\begin{aligned}
\mathbb{E}[\mathbf{X}] &= \underset{\substack{\uparrow \\ 1}}{\mathbb{E}[\mathbf{X} \mid \mathrm{S}]}\ \underset{\substack{\uparrow \\ p}}{\mathbb{P}[\mathrm{S}]} + \underset{\substack{\uparrow \\ 1+\mathbb{E}[\mathbf{X}]}}{\mathbb{E}[\mathbf{X} \mid \mathrm{F}]}\ \underset{\substack{\uparrow \\ (1-p)}}{\mathbb{P}[\mathrm{F}]} \\
&= p \cdot 1 + (1-p) \cdot (1 + \mathbb{E}[\mathbf{X}]) \\
&= 1 + (1-p) \cdot \mathbb{E}[\mathbf{X}].
\end{aligned}
$$

Solve for $\mathbb{E}[\mathbf{X}]$,

$$\mathbb{E}[\mathbf{X}] = \frac{1}{p}.$$

Practice. Exercise 6.5 and 6.8.