

Foundations of Computer Science

Lecture 22

Infinity

Size versus Cardinality: Comparing “Sizes”

Countable: Sets Which Are Not “Larger” Than \mathbb{N}

Is There A Set “Larger” Than \mathbb{N} ? Cantor’s Diagonal Argument

Infinity and Computing

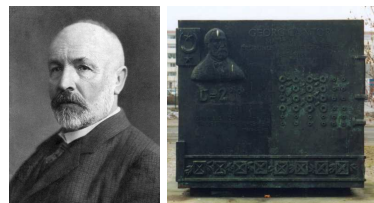


Our Short Stroll Through Discrete Math

- ➊ Precise statements, proofs and logic.
- ➋ **INDUCTION.**
- ➌ Recursively defined structures and Induction. (Data structures; PL)
- ➍ Sums and asymptotics. (Algorithm analysis)
- ➎ Number theory. (Cryptography; probability; fun)
- ➏ Graphs. (Relationships/conflicts; resource allocation; routing; scheduling, ...)
- ➐ Counting. (Enumeration and brute force algorithms)
- ➑ Probability. (Real world algorithms involve randomness/uncertainty)
 - ▶ Inputs arrive in a random order;
 - ▶ Randomized algorithms (primality testing, machine learning, routing, conflict resolution ...)
 - ▶ Expected value is a summary of what happens. Variance tells you how good the summary is.

Today: Infinity

- ➊ Comparing “sizes” of sets: countable.
 - Rationals are countable.
- ➋ Uncountable
 - Infinite binary strings.
- ➌ What does Infinity have to do with computing?



Georg Cantor

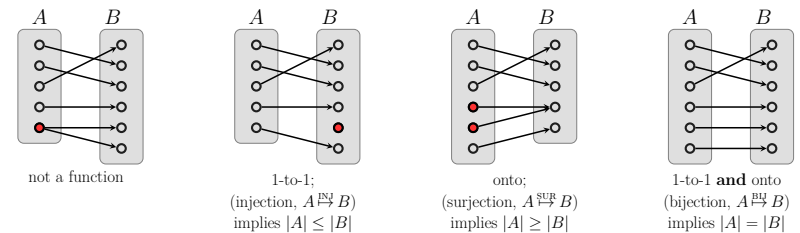
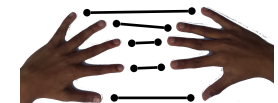
“Size” of a Set: Cardinality

You have **5** fingers on each hand.

You must know how to count.

You have an *equal* number of fingers on each hand.

All you need is a correspondence.



Cardinality $|A|$ (“size”), read “cardinality of A ,” is the number of elements for finite sets

$|A| \leq |B|$ iff there is an injection (1-to-1) from A to B , i.e., $f : A \rightarrow B$.

$|A| > |B|$ iff there is no injection from A to B .

$|A| \geq |B|$ iff there is a surjection (onto) from A to B , i.e., $f : A \rightarrow B$.

$|A| = |B|$ iff there is a bijection (1-to-1 and onto) from A to B , i.e., $f : A \rightarrow B$.

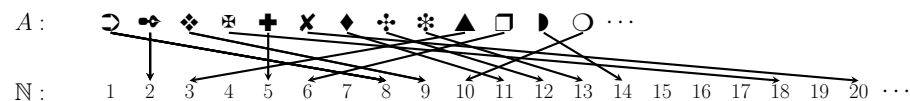
$|A| \leq |B|$ AND $|B| \leq |A| \rightarrow |A| = |B|$. (Cantor-Bernstein Theorem)

A Countable Set's Cardinality Is At Most $|\mathbb{N}|$

Finite sets : $|A| = n$ if and only if there is a bijection from A to $\{1, \dots, n\}$.

Infinite sets: The set A is countable if $|A| \leq |\mathbb{N}|$. A is "smaller than" \mathbb{N} .

To show that A is countable you must find a 1-to-1 mapping from A to \mathbb{N} .



You cannot skip over any elements of A , but you might not use every element of \mathbb{N} .

To prove that a function $f : A \mapsto \mathbb{N}$ is an injection:

- 1: Assume f is *not* an injection. (Proof by contradiction.)
- 2: This means there is a pair $x, y \in A$ for which $x \neq y$ and $f(x) = f(y)$.
- 3: Use $f(x) = f(y)$ to prove that $x = y$, a contradiction. Hence, f is an injection.

All Finite Sets are Countable

$A = \{3, 6, 8\}$. To show $|A| \leq \mathbb{N}$, we give an injection from A to \mathbb{N} ,

$$3 \mapsto 1 \quad 6 \mapsto 2 \quad 8 \mapsto 3.$$

For an arbitrary finite set $A = \{a_1, a_2, \dots, a_n\}$, \mathbb{N} ,

$$a_1 \mapsto 1 \quad a_2 \mapsto 2 \quad a_3 \mapsto 3 \quad \dots \quad a_n \mapsto n.$$

Non-negative integers $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ are countable

How can this be? \mathbb{N}_0 contains every element in \mathbb{N} *plus* 0?

To prove $|\mathbb{N}_0| \leq |\mathbb{N}|$, we give an injection $f : \mathbb{N}_0 \mapsto \mathbb{N}$,

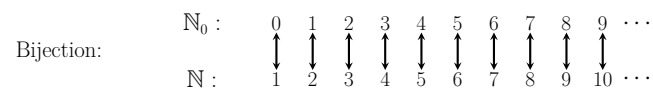
$$f(x) = x + 1, \quad \text{for } x \in \mathbb{N}_0.$$

Proof. Assume f is not an injection. So, there are $x \neq y$ in \mathbb{N}_0 with $f(x) = f(y)$:

$$x + 1 = f(x) = f(y) = y + 1.$$

That is $x + 1 = y + 1$ or $x = y$, which contradicts $x \neq y$. ■

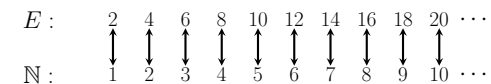
Also, $|\mathbb{N}| \leq |\mathbb{N}_0|$ because $\mathbb{N} \subseteq \mathbb{N}_0 \rightarrow |\mathbb{N}_0| = |\mathbb{N}|$. (Cantor-Bernstein)



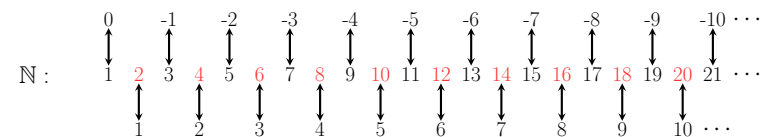
Positive Even Numbers and Integers are Countable

$E = \{2, 4, 6, \dots\}$. Surely $|E| = \frac{1}{2}|\mathbb{N}|$?

The bijection $f(x) = \frac{1}{2}x$ proves $|E| = |\mathbb{N}|$



$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$. $|\mathbb{Z}| = |\mathbb{N}|$.

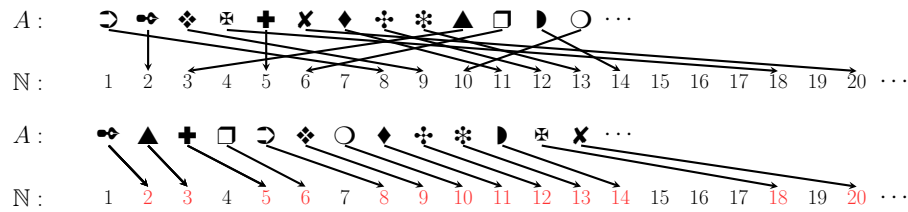


Exercise. What is a mathematical formula for the bijection?

Every Countable Set Can Be “Listed”

{3, 6, 8} is a list. $E = \{2, 4, 6, \dots\}$ is a list. What about \mathbb{Z} ?

$\dots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \dots$ ← not a list
 $0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \dots$ ← list



$\mathbb{N}_0 : \{0, 1, 2, 3, 4, 5, \dots\}$ $E : \{2, 4, 6, 8, 10, \dots\}$ $\mathbb{Z} : \{0, +1, -1, +2, -2, +3, -3, +4, -4, \dots\}$

- 1 Different elements are assigned to different list-positions.
- 2 Can determine the list-position of *any* element in the set. For \mathbb{Z} ,

$$\text{list position of } z = \begin{cases} 2z & z > 0; \\ 2|z| + 1 & z \leq 0; \end{cases}$$

Union of Two Countable Sets is Countable

A and B are countable, so they can be listed.

$$A = \{a_1, a_2, a_3, a_4, a_5, \dots\} \quad B = \{b_1, b_2, b_3, b_4, b_5, \dots\}$$

Here is a list for $A \cup B$

$$A \cup B = \{a_1, a_2, a_3, a_4, a_5, \dots, b_1, b_2, b_3, b_4, b_5, \dots\} \quad \times$$

What is the list-position of b_1 ? Cannot use “...” twice.

$$A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, a_5, b_5, \dots\}$$

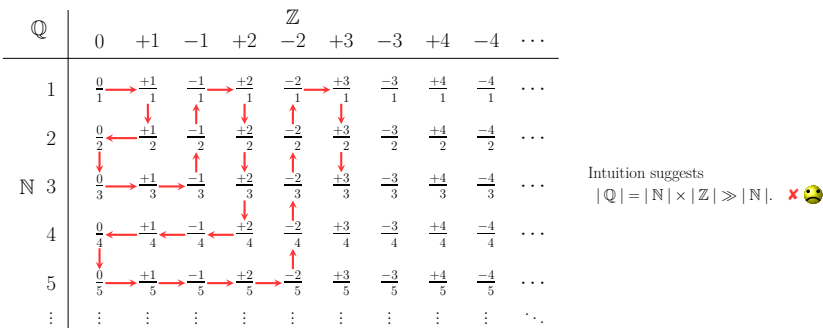
list-position of a_i is $2i - 1$;

list-position of b_i is $2i$.

Pop Quiz. Get a list of \mathbb{Z} with $A = \{0, -1, -2, -3, \dots\}$ and $B = \{1, 2, 3, \dots\}$ using union.

Rationals are Countable: $|\mathbb{Q}| = |\mathbb{N}|$

This is surprising because between any two rationals there is another (not true for \mathbb{N}).



Intuition suggests
 $|\mathbb{Q}| = |\mathbb{N}| \times |\mathbb{Z}| \gg |\mathbb{N}|$. \times 😞

$$\mathbb{Q} = \left\{ \frac{0}{1}, \frac{+1}{1}, \frac{-1}{1}, \frac{+2}{1}, \frac{-2}{1}, \frac{+3}{1}, \frac{-3}{1}, \frac{+4}{1}, \frac{-4}{1}, \dots \right\}$$

$$|\{\text{Rational Values}\}| \leq |\mathbb{Q}| \leq |\mathbb{N}|$$

Exercise. What is a mathematical formula for the list-position of $z/n \in \mathbb{Q}$?

Programs are Countable

Programs are finite binary strings. We show that all finite binary strings \mathcal{B} are countable.

$$\mathcal{B} = \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, 0000, \dots\} \quad \leftarrow \text{list}$$

Pop Quiz. What is the list-position of 0110?

Exercise. For the $(k+1)$ -bit string $b = b_k b_{k-1} \dots b_1 b_0$, define the strings numerical value:

$$\text{value}(b) = b_0 \cdot 2^0 + b_1 \cdot 2^1 + \dots + b_{k-1} \cdot 2^{k-1} + b_k \cdot 2^k$$

Show:

$$\text{list-position of } b = 2^{\text{length}(b)} + \text{value}(b)$$

$\mathbb{N}_0, E, \mathbb{Z}, \mathbb{Q}, \mathcal{B}$ are countable, ... **Is Everything Countable?**

Infinite Binary Strings are Uncountable

Cantor's Diagonal Argument: Assume there is a list of *all* infinite binary strings.

b_1 : 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 ...
 b_2 : 0 0 1 1 0 1 0 0 1 0 0 0 0 0 1 0 ...
 b_3 : 1 1 0 0 0 0 1 0 0 0 0 1 1 0 0 ...
 b_4 : 1 0 1 0 0 1 0 0 0 0 1 0 0 0 0 ...
 b_5 : 0 1 1 0 1 0 1 0 1 0 0 0 0 0 0 ...
 b_6 : 0 1 0 1 1 0 0 0 0 1 0 0 0 0 0 ...
 b_7 : 0 0 1 0 0 1 0 0 0 0 0 0 0 0 1 ...
 b_8 : 0 0 1 0 1 1 0 1 0 0 0 0 1 0 0 ...
 b_9 : 0 0 0 0 1 1 0 0 0 0 0 1 0 0 0 ...
 b_{10} : 1 0 1 1 1 0 1 0 0 0 1 1 0 0 0 ...
⋮

Consider the "diagonal string"

$b = 0000100101\dots$

Flip the bits,

$\bar{b} = 1111011010\dots$

\bar{b} is not in the list (differs in the i th position from b_i), a contradiction.

Reals are Uncountable

Every real has an infinite binary representation and every infinite binary string evaluates to a real number.

e.g. $0.0011111111111111\dots = \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \dots = \frac{1}{2}$.

That is $|\{\text{reals in } [0, 1]\}| = |\{\text{infinite binary strings}\}| > |\mathbb{N}|$.

Infinity and Computing

Cantor took on the abstract beast Infinity. (1874)

~ 60 years later, Alan Turing asked the abstract question: What can we compute? (1936)

Every binary function f on \mathbb{N} corresponds to a infinite binary string $f(1)f(2)f(3)\dots$,

n :	1	2	3	4	5	6	7	8	9	10	...
$f(n)$:	0	1	1	0	1	0	0	0	1	1	...

Every program is a finite binary string. For example,

```
int main(); //a program that does nothing
```

is the finite binary string (ASCII code)

```
011010010110111001110100001000001101101011000010110100101101110001010000010100100111011
```

Programs \leftarrow Countable

Functions \leftarrow Uncountable

\rightarrow

$|\{\text{functions on } \mathbb{N}\}| \gg |\{\text{programs}\}|$

There are MANY MANY functions that cannot be computed by programs!
Are there interesting, useful functions that cannot be computed by programs?