Learning From Data
Lecture 4
Real Learning is Feasible

Real Learning vs. Verification
The Two Step Solution to Learning
Closer to Reality: Error and Noise

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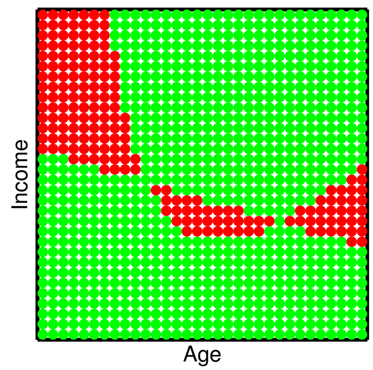
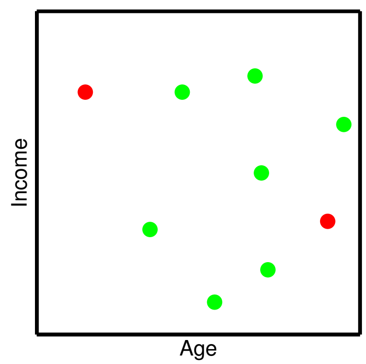
Verification

h



$E_{\text{out}}(h)$

$\Big\downarrow \mathcal{D}$



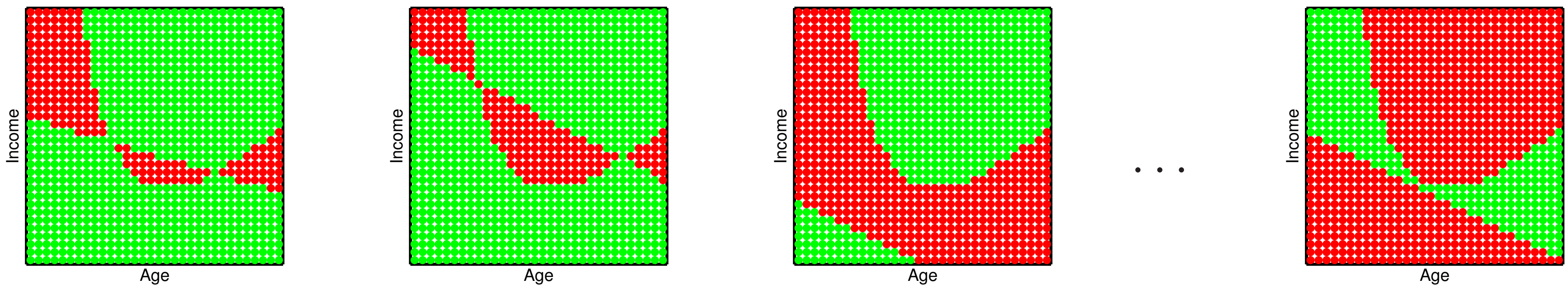
$E_{\text{in}}(h) = \frac{2}{9}$

Hoeffding: $E_{\text{out}}(h) \approx E_{\text{in}}(h)$ (with high probability)

$$\mathbb{P}[|E_{\text{in}}(h) - E_{\text{out}}(h)| > \epsilon] \leq 2e^{-2N\epsilon^2}.$$

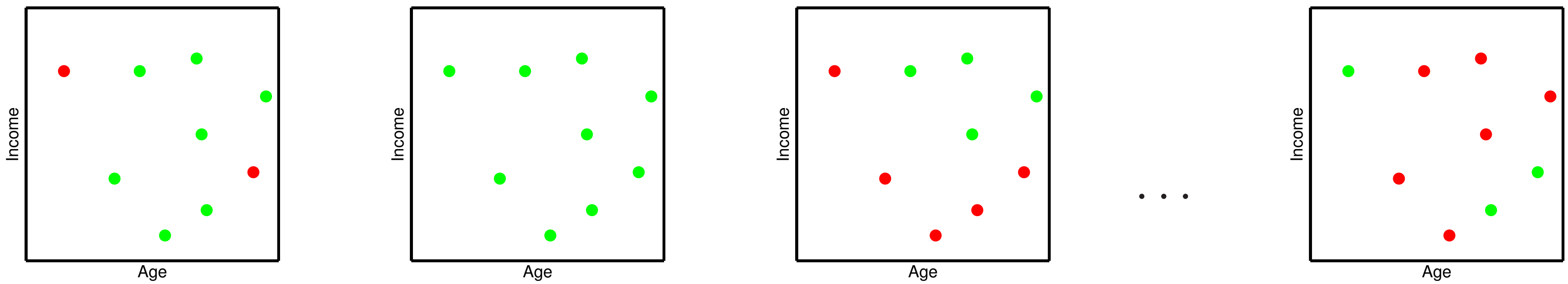
Real Learning – Finite Learning Models

h_1 $\quad$ h_2 $\quad$ h_3 $\quad \cdots \quad$ h_M



$E_{\text{out}}(h_1)$ $\quad$ $E_{\text{out}}(h_2)$ $\quad$ $E_{\text{out}}(h_3)$ $\quad$ $E_{\text{out}}(h_M)$

$$\downarrow \mathcal{D}$$

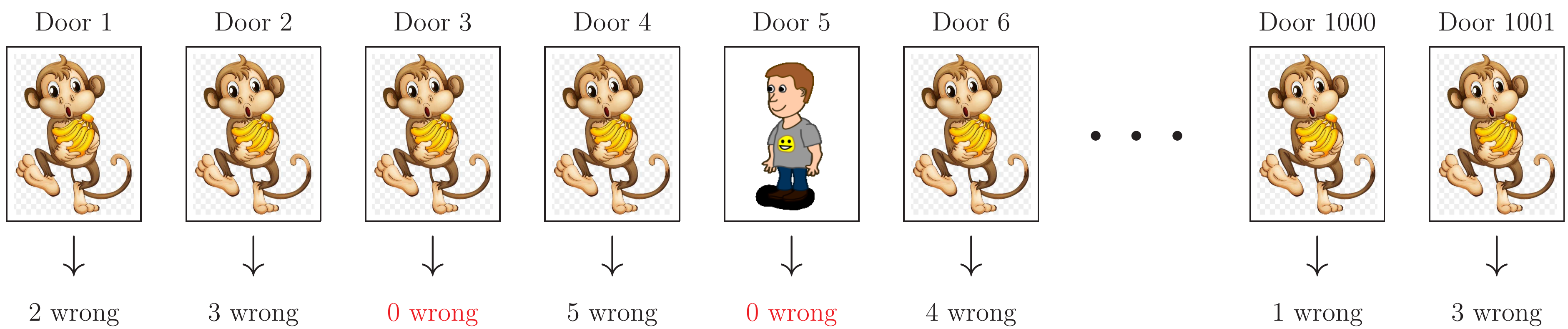


$E_{\text{in}}(h_1) = \frac{2}{9}$ $\quad$ $E_{\text{in}}(h_2) = \mathbf{0}$ $\quad$ $E_{\text{in}}(h_3) = \frac{5}{9}$ $\quad$ $E_{\text{in}}(h_M) = \frac{6}{9}$

Pick the hypothesis with minimum E_{in}; will E_{out} be small?

1000 Monkeys Behind Closed Doors

5-question A/B test. Monkeys answer randomly. Child gets all right.



- What are your chances of picking the child?
- What can you do about it? (You can't peek behind the door. ☹)

More Monkeys: E_{in} Can't Reach Out to E_{out}.

Selection Bias Illustrated with Coins

Coin tossing example:

- If we toss one coin and get no HEADS, its very surprising. $\mathbb{P} = \frac{1}{2^N}$

  We expect it is biased: $\mathbb{P}[\text{heads}] \approx 0$.

- Tossing 70 coins, and *find one* with no heads. Is it surprising? $\mathbb{P} = 1 - \left(1 - \frac{1}{2^N}\right)^{70}$

  Do we expect $\mathbb{P}[\text{heads}] \approx 0$ for the selected coin?

  Similar to the "birthday problem": among 30 people, two will likely share the same birthday.

- This is called *selection bias.*

  Selection bias is a very serious trap. For example medical screening.

$$\boxed{\textbf{Search Causes Selection Bias}}$$

Hoeffding says that $E_{\text{in}}(g) \approx E_{\text{out}}(g)$ for Finite $\mathcal{H}$

$$\mathbb{P}\left[|E_{\text{in}}(g) - E_{\text{out}}(g)| > \epsilon\right] \le 2|\mathcal{H}|e^{-2\epsilon^2 N}, \qquad \text{for any } \epsilon > 0.$$

$$\mathbb{P}\left[|E_{\text{in}}(g) - E_{\text{out}}(g)| \le \epsilon\right] \ge 1 - 2|\mathcal{H}|e^{-2\epsilon^2 N}, \qquad \text{for any } \epsilon > 0.$$

We don't care how g was obtained, *as long as it is from* $\mathcal{H}$

Some Basic Probability
Events A, B

Implication
If $A \implies B$ ($A \subseteq B$) then $\mathbb{P}[A] \le \mathbb{P}[B]$.

Union Bound
$\mathbb{P}[A \text{ or } B] = \mathbb{P}[A \cup B] \le \mathbb{P}[A] + \mathbb{P}[B]$.

Bayes' Rule
$\mathbb{P}[A|B] = \dfrac{\mathbb{P}[B|A] \cdot \mathbb{P}[A]}{\mathbb{P}[B]}$

Proof: Let $M = |\mathcal{H}|$.

The event "$|E_{\text{in}}(g) - E_{\text{out}}(g)| > \epsilon$" implies
"$|E_{\text{in}}(h_1) - E_{\text{out}}(h_1)| > \epsilon$" OR ... OR "$|E_{\text{in}}(h_M) - E_{\text{out}}(h_M)| > \epsilon$"

So, by the implication and union bounds:

$$\begin{aligned}
\mathbb{P}[|E_{\text{in}}(g) - E_{\text{out}}(g)| > \epsilon] &\le \mathbb{P}\left[\operatorname*{OR}_{m=1}^{M} |E_{\text{in}}(h_M) - E_{\text{out}}(h_M)| > \epsilon\right] \\
&\le \sum_{m=1}^{M} \mathbb{P}[|E_{\text{in}}(h_m) - E_{\text{out}}(h_m)| > \epsilon], \\
&\le 2Me^{-2\epsilon^2 N}.
\end{aligned}$$

(The last inequality is because we can apply the Hoeffding bound to each summand)

Interpreting the Hoeffding Bound for Finite $|\mathcal{H}|$

$$\mathbb{P}\left[|E_{\text{in}}(g) - E_{\text{out}}(g)| > \epsilon\right] \le 2|\mathcal{H}|e^{-2\epsilon^2 N}, \qquad \text{for any } \epsilon > 0.$$

$$\mathbb{P}\left[|E_{\text{in}}(g) - E_{\text{out}}(g)| \le \epsilon\right] \ge 1 - 2|\mathcal{H}|e^{-2\epsilon^2 N}, \qquad \text{for any } \epsilon > 0.$$

Theorem. With probability at least $1 - \delta$,

$$E_{\text{out}}(g) \le E_{\text{in}}(g) + \sqrt{\frac{1}{2N} \log \frac{2|\mathcal{H}|}{\delta}}.$$

We don't care how g was obtained, *as long as* $g \in \mathcal{H}$

Proof: Let $\delta = 2|\mathcal{H}|e^{-2\epsilon^2 N}$. Then

$$\mathbb{P}\left[|E_{\text{in}}(g) - E_{\text{out}}(g)| \le \epsilon\right] \ge 1 - \delta.$$

In words, with probability at least $1 - \delta$,

$$|E_{\text{in}}(g) - E_{\text{out}}(g)| \le \epsilon.$$

This implies

$$E_{\text{out}}(g) \le E_{\text{in}}(g) + \epsilon.$$

From the definition of δ, solve for ϵ:

$$\epsilon = \sqrt{\frac{1}{2N} \log \frac{2|\mathcal{H}|}{\delta}}.$$

E_{in} Reaches Outside to E_{out} when $|\mathcal{H}|$ is Small

$$E_{\text{out}}(g) \leq E_{\text{in}}(g) + \sqrt{\frac{1}{2N} \log \frac{2|\mathcal{H}|}{\delta}}.$$

If $N \gg \ln |\mathcal{H}|$, then $E_{\text{out}}(g) \approx E_{\text{in}}(g)$.

- Does not depend on $\mathcal{X}$, $P(\mathbf{x})$, f or how g is found.
- Only requires $P(\mathbf{x})$ to generate the data points independently *and also* the test point.

What about $E_{\text{out}} \approx 0$?

The 2 Step Approach to Getting $E_{\text{out}} \approx 0$:

$$\boxed{\begin{array}{ll} (1) & E_{\text{out}}(g) \approx E_{\text{in}}(g). \\ (2) & E_{\text{in}}(g) \approx 0. \end{array}}$$

Together, these ensure $E_{\text{out}} \approx 0$.

How to verify (1) since we do not know E_{out}

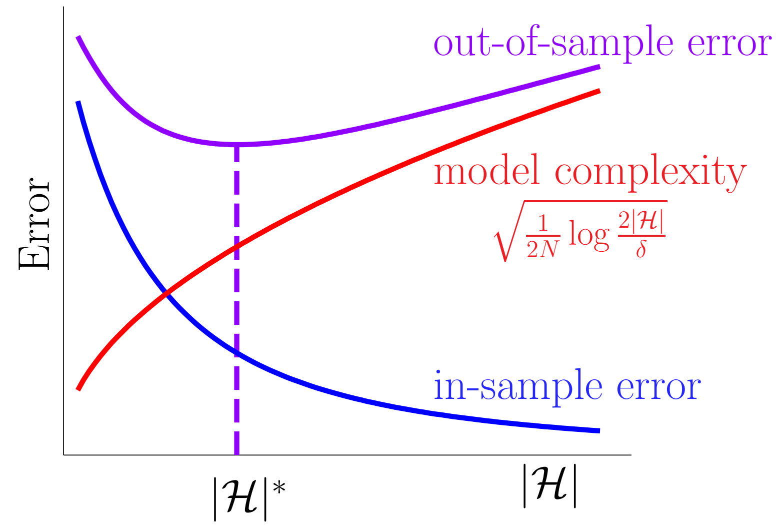
– must *ensure* it theoretically - Hoeffding.

We can ensure (2) (for example PLA)

– modulo that we can guarantee (1)

There is a tradeoff:

- Small $|\mathcal{H}| \implies E_{\text{in}} \approx E_{\text{out}}$
- Large $|\mathcal{H}| \implies E_{\text{in}} \approx 0$ is more likely.



Feasibility of Learning (Finite Models)

- No Free Lunch: can't know anything outside $\mathcal{D}$, *for sure.*

- Can "learn" with high probability if $\mathcal{D}$ is *i.i.d.* from $P(\mathbf{x})$.
  $E_{\text{out}} \approx E_{\text{in}}$ (E_{in} can reach outside the data set to E_{out}).

- **We want $E_{\text{out}} \approx 0$.**

- The two step solution. We trade $E_{\text{out}} \approx 0$ for 2 goals:
  (i) $E_{\text{out}} \approx E_{\text{in}}$;
  (ii) $E_{\text{in}} \approx 0$.
  We know E_{in}, not E_{out}, but we can *ensure* (*i*) if $|\mathcal{H}|$ is small.

This is a big step!

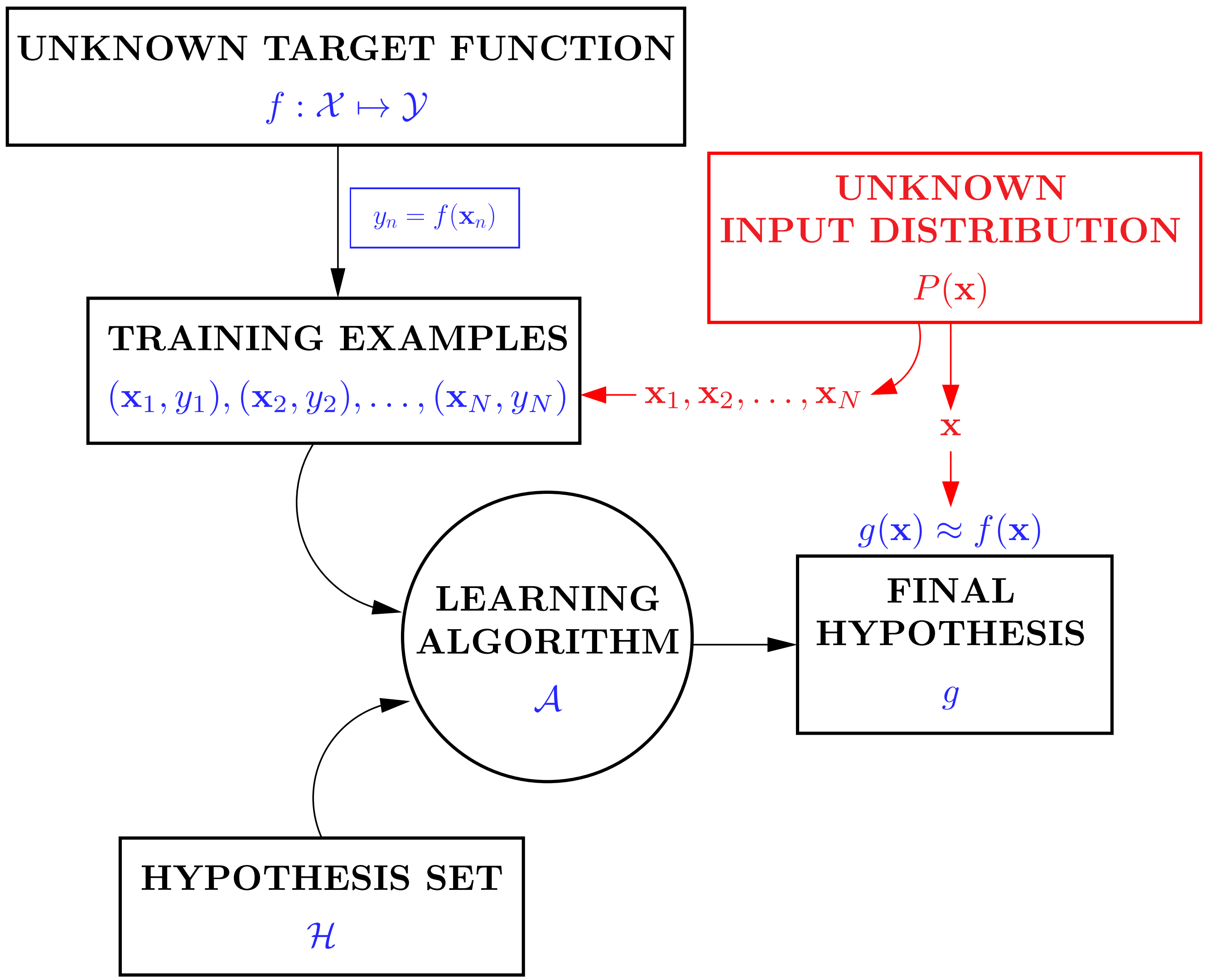
- What about infinite $\mathcal{H}$ - the perceptron?

"Complex" Target Functions are Harder to Learn

What happened to the "difficulty" (complexity) of f?

- Simple $f \implies$ can use small $\mathcal{H}$ to get $E_{\text{in}} \approx 0$ (need smaller N).
- Complex $f \implies$ need large $\mathcal{H}$ to get $E_{\text{in}} \approx 0$ (need larger N).

Revising the Learning Problem – Adding in Probability

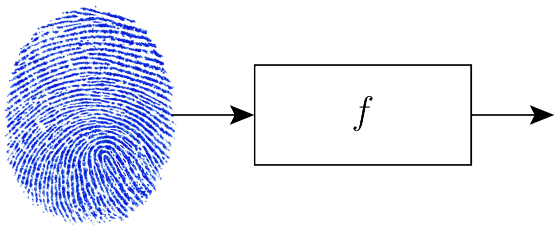


Error and Noise

Error Measure: How to quantify that $h \approx f$.

Noise: $y_n \neq f(\mathbf{x}_n)$.

Finger Print Recognition



$$\begin{cases} +1 & \text{you} \\ -1 & \text{intruder} \end{cases}$$

Two types of error.

		f: $+1$	f: -1
h	$+1$	no error	**false accept**
	-1	**false reject**	no error

In any application you need to think about how to penalize each type of error.

		f: $+1$	f: -1
h	$+1$	0	1
	-1	**10**	0

Supermarket

		f: $+1$	f: -1
h	$+1$	0	**1000**
	-1	1	0

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Take Away

Error measure is specified by the user.

If not, choose one that is
- *plausible (conceptually appealing)*
- *friendly (practically appealing)*

Almost All Error Mearures are Pointwise

Compare h and f on individual points $\mathbf{x}$ using a pointwise error $e(h(\mathbf{x}), f(\mathbf{x}))$:

$$\text{Binary error:} \quad e(h(\mathbf{x}), f(\mathbf{x})) = [\![h(\mathbf{x}) \neq f(\mathbf{x})]\!] \qquad \text{(classification)}$$

$$\text{Squared error:} \quad e(h(\mathbf{x}), f(\mathbf{x})) = (h(\mathbf{x}) - f(\mathbf{x}))^2 \qquad \text{(regression)}$$

In-sample error:

$$E_{\text{in}}(h) = \frac{1}{N} \sum_{n=1}^{N} e(h(\mathbf{x}_n), f(\mathbf{x}_n)).$$

Out-of-sample error:

$$E_{\text{out}}(h) = \mathbb{E}_{\mathbf{x}}[e(h(\mathbf{x}), f(\mathbf{x}))].$$

Noisy Targets

age	32 years
gender	male
salary	40,000
debt	26,000
years in job	1 year
years at home	3 years
...	...

Approve for credit?

Consider two customers with the **same** credit data.
They can have **different** behaviors.

The target 'function' is not a deterministic function but a *stochastic* function.

$$\text{`}f(\mathbf{x})\text{'} = P(y|\mathbf{x})$$

Learning Setup with Error Measure and Noisy Targets