FINAL: 180 Minutes

Last Name: ____________________

First Name: ____________________

RIN: ____________________

Section: ____________________

Answer **ALL** questions. You may use **two** single sided $8\frac{1}{2} \times 11$ crib sheets.

NO COLLABORATION or electronic devices. Any violations result in an F.

NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

1	2	3	4	5	6	Total
100	50	50	50	50	50	350

1 Circle at most one answer per question. 10 points for each correct answer and -5 points for each incorrect answer (blank answer is 0 points). Don't guess!

(a) $P(n)$ is a predicate ($n \in \mathbb{N}$). $P(1), P(2), P(3)$ are true, and $P(n) \to P(n+4)$ is true for $n \geq 1$. For which n can we be **sure** $P(n)$ is true?

$\boxed{\text{A}}$ All $n \geq 1$ except multiples of 2.

$\boxed{\text{B}}$ All $n \geq 1$ except multiples of 4.

$\boxed{\text{C}}$ All $n \geq 1$

$\boxed{\text{D}}$ Only $n = 1, 2, 3$.

(b) Of the following five sets, list *all* that are countable ($\mathcal{A}$ is countable if $\mathbb{N} \overset{\text{surj}}{\longrightarrow} \mathcal{A}$):
(I) Prime numbers; (II) Rational numbers; (III) Integers; (IV) Even numbers; (V) Infinite binary strings.

$\boxed{\text{A}}$ I and III.

$\boxed{\text{B}}$ I and II and III and IV.

$\boxed{\text{C}}$ I and III and V.

$\boxed{\text{D}}$ II and III and IV.

(c) A class with 25 students needs to choose a representative committee which is a subset of 5 students. How many different committees can be formed?

$\boxed{\text{A}}$ 25^5.

$\boxed{\text{B}}$ $\dfrac{25!}{20! \times 5!}$.

$\boxed{\text{C}}$ $\dfrac{25!}{5!}$.

$\boxed{\text{D}}$ $25 \times 24 \times 23 \times 22 \times 21 = \dfrac{25!}{20!}$.

(d) A friendship network has 7 people and each person has at least 1 friend. 6 of the people have *exactly two friends*. How many friends can the 7th person have? Give all possibilities.

$\boxed{\text{A}}$ The seventh person could have either 2 or 4 friends.

$\boxed{\text{B}}$ The seventh person could have either 2 or 4 or 6 friends.

$\boxed{\text{C}}$ The seventh person could have either 1 or 2 or 3 friends.

$\boxed{\text{D}}$ The seventh person could have any number of friends that is greater than 1.

(e) Compute the summation $(0+1) + (1+2) + (2+4) + (3+8) + \cdots + (10 + 2^{10}) = \sum_{i=0}^{10}(i + 2^i)$

$\boxed{\text{A}}$ 2048.

$\boxed{\text{B}}$ 2102.

$\boxed{\text{C}}$ 1078.

$\boxed{\text{D}}$ 2200.

(f) You have a known fact that $0 = 0$ and all the standard operations of algebra you learned in high-school math. Which of the following is a valid proof that $7 = 7$:

$$
\begin{array}{llcl}
 & & \underline{\text{I}} & \\
1. & 7 & = & 7 \\
2. & 7-7 & = & 7-7 \\
3. & 0 & = & 0 \quad \checkmark \\
\hline
\rightarrow & 7 & = & 7
\end{array}
\qquad
\begin{array}{llcl}
 & & \underline{\text{II}} & \\
1. & 7 & \neq & 7 \\
2. & 7-7 & \neq & 7-7 \\
3. & 0 & \neq & 0 \quad \textbf{!FISHY} \\
\hline
\rightarrow & 7 & = & 7
\end{array}
\qquad
\begin{array}{llcl}
 & & \underline{\text{III}} & \\
1. & 0 & = & 0 \\
2. & 0+7 & = & 0+7 \\
3. & 7 & = & 7 \quad \checkmark \\
\hline
\rightarrow & 7 & = & 7
\end{array}
$$

$\boxed{\text{A}}$ I & II & III. $\boxed{\text{B}}$ II & III. $\boxed{\text{C}}$ I & II $\boxed{\text{D}}$ I & III.

(g) Let $f(n) = \sum_{i=1}^{n} i$ and $g(n) = 2^{3\log_2 n}$. What is the big-Oh relationship between f and g?

$\boxed{\text{A}}$ $f(n) = O(g(n))$ and $g(n) = O(f(n))$.

$\boxed{\text{B}}$ $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

$\boxed{\text{C}}$ $f(n) \neq O(g(n))$ and $g(n) = O(f(n))$.

$\boxed{\text{D}}$ $f(n) \neq O(g(n))$ and $g(n) \neq O(f(n))$.

(h) You independently generate the ten bits of a binary sequence $b_1 b_2 \cdots b_{10}$ with $\mathbb{P}[b_i = 0] = \frac{1}{2}$. Compute the probability that the sequence is sorted from low to high. For example 0000111111 is sorted.

$\boxed{\text{A}}$ $\dfrac{10}{1024}$

$\boxed{\text{B}}$ $\dfrac{11}{1024}$

$\boxed{\text{C}}$ $\dfrac{20}{1024}$

$\boxed{\text{D}}$ $\dfrac{12}{1024}$

(i) x_1, x_2, x_3 are non-negative integers. Compute the number of different solutions to $x_1 + x_2 + x_3 = 100$. (For example two different solutions are $1 + 2 + 97 = 100$ and $97 + 1 + 2 = 100$.)

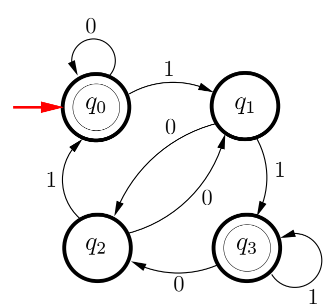
$\boxed{\text{A}}$ 10302

$\boxed{\text{B}}$ 5151

$\boxed{\text{C}}$ 4949

$\boxed{\text{D}}$ 5050

(j) For the automaton on the right, which input string is accepted? (Strings are processed from left to right.)



$\boxed{\text{A}}$ 010101

$\boxed{\text{B}}$ 0101011

$\boxed{\text{C}}$ 01010110

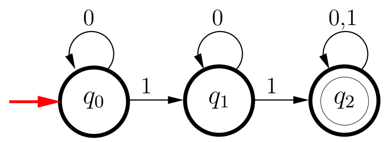
$\boxed{\text{D}}$ 010101100

2 Proofs

1. Prove that for all integers $n \geq 1$: $\qquad n2^n \leq 3^n$

2. Prove that *every* odd natural number is the difference of two square numbers.

3 Finite Automaton with a Random Input String



The automaton to the right processes a random binary string $b_1 b_2 \ldots b_n$ of length n generated as follows: you independently generate each bit b_i with $\mathbb{P}[b_i = 1] = p$ and $\mathbb{P}[b_i = 0] = 1 - p$. Show that the probability that the string is accepted is

$$\mathbb{P}[\text{ random input string is accepted }] = 1 - (1-p)^n - np(1-p)^{n-1}.$$

[Hints: (i) Figure out a simple property of a string for it to be accepted. (ii) Binomial distribution.]

4 Probability and Expectation

(a) You independently roll 3 fair dice D_1, D_2, D_3 and let $S = D_1 + D_2 + D_3$ be the sum. Compute:

(i) $\mathbb{P}[S = 8]$

(ii) $\mathbb{P}[S = 8 \mid D_1 = 1]$

(iii) Compute the expectation and variance of S.

(b) You toss a fair coin independently until you get two heads *in a row*. Let X be the number of tosses. Compute $\mathbb{E}[X]$ using the law of total expectation:

(i) Consider the 3 cases T, HT, HH for how the tosses may start and show that

$$\mathbb{E}[X] = \tfrac{1}{2}(1 + \mathbb{E}[X]) + \tfrac{1}{4}(2 + \mathbb{E}[X]) + \tfrac{1}{2}.$$

(ii) Use (i) to show that $\mathbb{E}[X] = 6$.

5 Context Free Grammars

This problem is about the language $\mathcal{L}$ generated by the CFG:

$$\begin{aligned} S &\rightarrow 1T \mid 0T \\ T &\rightarrow 1T1 \mid 0T0 \mid \epsilon \end{aligned}$$

(a) Is the string 1010010 in $\mathcal{L}$? If yes then give a derivation or parse tree; if no then explain why.

(b) Prove that the length of every string in $\mathcal{L}$ is odd.

6 Turing Machine

(a) What is the difference between a Turing-recognizable language and a Turing-decidable language?

(b) Consider the arithmetic task of squaring, which corresponds to the language $\mathcal{L} = \{0^n\#0^{n^2} | n \geq 1\}$.

(i) Circle the simplest model of computing that you think solves the problem $\mathcal{L}$:

Finite Automaton   Context Free Grammar   Turing Machine

(ii) Give your machine from (i) that solves $\mathcal{L}$ (for a TM, a high level description will do).

SCRATCH

SCRATCH