FINAL: 180 Minutes

Last Name: ____________________

First Name: ____________________

RIN: ____________________

Section: ____________________

Answer **ALL** questions. You may use **two** double sided $8\frac{1}{2} \times 11$ crib sheets.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

1	2	3	4	5	Total
150	50	50	50	50	350

1 Circle at most one answer per question. 10 points for each correct answer.

(1) The **negation** of "All Malik's friends are big and strong" is

A None of Malik's friends are big and strong.

B Malik has a friend who is either small or weak (or both).

C All Malik's friends are small and weak.

D All Malik's friends are either small or weak (or both).

E Malik has no friends who are small or weak.

(2) What is the *most accurate* order relation between $3^{\log_2 n}$ and n^2?

A $3^{\log_2 n} \in o(n^2)$.

B $3^{\log_2 n} \in O(n^2)$.

C $3^{\log_2 n} \in \Theta(n^2)$.

D $3^{\log_2 n} \in \Omega(n^2)$.

E $3^{\log_2 n} \in \omega(n^2)$.

(3) Compute the summation $\sum_{i=1}^{20} (-1)^i i^2$

A 190.

B 200.

C 210.

D 220.

E 230.

(4) Let $f(n) = \sum_{i=1}^{n} i$ and $g(n) = 4^{\log_2 n}$. What is the *most accurate* order relationship between f and g?

A $f \in o(g)$.

B $f \in O(g)$.

C $f \in \Theta(g)$.

D $f \in \Omega(g)$.

E $f \in \omega(g)$.

(5) Let $f(n)$ be a function satisfying the recurrence $f(0) = 0$; $f(n) = f(n-1) + \sqrt{n}$. Which order relationship describes f.

A $f \in \Theta(n)$.

B $f \in \Theta(n \log n)$.

C $f \in \Theta(n\sqrt{n})$.

D $f \in \Theta(n^2)$.

E $f \in \Theta(n^3)$.

(6) A class with 10 students needs to choose a president, vice-president and secretary (a student *cannot* fill multiple roles). In how many ways can this be done?

A 1000.

B 720.

C 120.

D 10!

E $\binom{10}{3}$.

(7) A fraternity orders 5 pizzas (eg. 2 with sausage and 3 with meatballs & onion). There are 5 toppings. A pizza can have 0,1 or 2 toppings. How many ways are there for the fraternity to make its order?

A 16.

B 16^5.

C $\binom{16}{5}$.

D $\binom{20}{15}$.

E $16 \times 15 \times 14 \times 13 \times 12$.

(8) A friendship network has 6 people Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ. If you add up the number of friends of each person, you get a total of 26. How many *different* social network graphs could correspond to this friendship network. (Two graphs are different if they don't have exactly the same edges.)

A 0.

B 95.

C 105.

D 115.

E 125.

(9) You are thinking of a graph with 5 nodes Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ. *Approximately* how many such graphs are there?

A 100.

B 500.

C 1000.

D 5000.

E 10000.

(10) X and Y are random variables (not necessarily independent). Which of the following is an expression for $Var(X+Y)$ (variance of the sum)?

A $Var(X) + Var(Y)$.

B $\mathbb{E}[(X+Y)^2]$.

C $\mathbb{E}[X^2] + \mathbb{E}[Y^2] - \mathbb{E}[X]^2 - \mathbb{E}[Y]^2$.

D $Var(X) + Var(Y) + 2\,\mathbb{E}[XY] - 2\,\mathbb{E}[X]\,\mathbb{E}[Y]$.

E $Var(X) + Var(Y) - 2Var(XY)$.

(11) You independently generate two random ten bit binary sequences and compute a new sequence using the BITWISE-OR of the two random sequences (treating 0 as FALSE and 1 as TRUE). Let X be the number of 1s in the result. What is $\mathbb{E}[X]$. (for example, 0001110010 BITWISE-OR 1000111000 = 1001111010.)

A 2.5

B 3.5

C 5

D 6.5

E 7.5

(12) About 1 in a 1000 people have Coeliac disease. The outcome of a test for Coeliac is random: the test makes a mistake on 1 in 10 people who have it (90% accuracy if you have Coeliac); the test makes a mistake on 1 in 100 people who do not have it (99% accuracy if you do not have Coeliac). You got tested, and the result was positive. *Approximately* what are the chances that you have Coeliac?

A 0.1%

B 10%

C 40%

D 80%

E 90%

(13) Which set is *not countable*?

A {1,3,5,7}.

B The prime numbers {2,3,5,7,...}.

C All possible angles between 0 and 360.

D All even numbers which are not a sum of two primes.

E All possible pairs of integers, $\mathbb{Z}^2$.

(14) A random binary string $b_1 b_2 \dots b_{10}$ of length 10 is the input to the automaton.

What is the probability that the string is accepted?

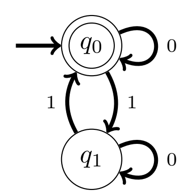
A 0.25

B 0.4

C 0.5

D 0.6

E 0.75



(15) Which string below is *not* in the language of the CFG: $S \longrightarrow \varepsilon \mid 0S|S0|11S$

A ε

B 1111

C 11011

D 0011000

E 001010

2 Positive Integer Partitions

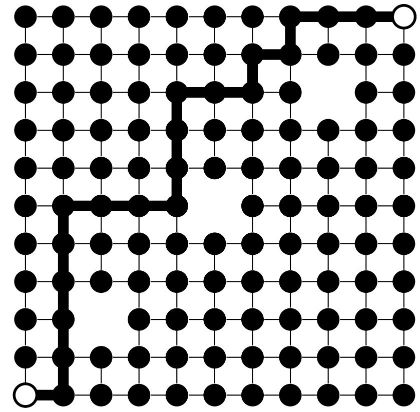
A positive partition of n is a *sequence* of *positive* integers that add up to n. For example, $(6, 4)$, $(4, 6)$ and $(2, 4, 2, 2)$ are different partitions of 10. How many positive partitions of n are there? Prove your answer.

3 Proofs

(a) *Prove* that $n^2 \leq 3^n$ for integer $n \geq 0$.

(b) *Prove* that $n^3 \notin O(n^2)$. You must *prove* that there is no *constant* C for which $n^3 \leq Cn^2$ for all $n \geq 1$.

4 Counting Paths on Graphs with Holes



A grid is missing nodes at $(2, 2)$, $(5, 5)$ and $(8, 8)$. A *shortest* path from the bottom left node (0,0) to the top right node (10,10) is shown.

How many different shortest paths go from (0,0) to (10,10)? (Two paths are different if they do not have exactly the same edges).

You may leave your answer in the form of a combination of binomial coefficients – you do not need to compute a numerical answer.

5 Turing Machine and Exponentiation

(a) *Prove*: the problem (language) $\mathcal{L} = \{0^n\#1^{2^n} \mid n \geq 1\}$ *cannot* be solved (accepted) by a finite automaton.

(b) Give a high-level description of a Turing Machine that solves $\mathcal{L} = \{0^n\#1^{2^n} \mid n \geq 1\}$.

SCRATCH

SCRATCH