FINAL: 180 Minutes

Last Name: Solutions

First Name: ______________________

RIN: ______________________

Section: ______________________

Answer **ALL** questions. You may use **two** double sided $8\frac{1}{2} \times 11$ crib sheets.

NO COLLABORATION or electronic devices. Any violations result in an F.

NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

1	2	3	4	5	Total
150	50	50	50	50	350

1 Circle at most one answer per question. 10 points for each correct answer.

(1) The negation of "All Malik's friends are big and strong" is

$\boxed{A}$ None of Malik's friends are big and strong.

$\boxed{B}$ Malik has a friend who is either small or weak (or both).

$\boxed{C}$ All Malik's friends are small and weak.

$\boxed{D}$ All Malik's friends are either small or weak (or both).

$\boxed{E}$ Malik has no friends who are small or weak.

B

$\sim p \wedge q \equiv \sim p \vee \sim q.$

$\neg \forall x\, P(x) \equiv \exists x\ \neg P(x).$

(2) What is the *most accurate* order relation between $3^{\log_2 n}$ and n^2?

$\boxed{A}$ $3^{\log_2 n} \in o(n^2)$.

$\boxed{B}$ $3^{\log_2 n} \in O(n^2)$.

$\boxed{C}$ $3^{\log_2 n} \in \Theta(n^2)$.

$\boxed{D}$ $3^{\log_2 n} \in \Omega(n^2)$.

$\boxed{E}$ $3^{\log_2 n} \in \omega(n^2)$.

A

$3^{\log_2 n} = (2^{\log_2 3})^{\log_2 n}$

$= 2^{\log_2 n \log_2 3} = 2^{\log_2 n^{\log_2 3}}$

$= n^{\log_2 3}$ $\quad 1 < \log_2 3 < 2$

$\therefore n^{\log_2 3} \in o(n^2)$

(3) Compute the summation $\sum_{i=1}^{20}(-1)^i i^2$

$\boxed{A}$ 190.

$\boxed{B}$ 200.

$\boxed{C}$ 210.

$\boxed{D}$ 220.

$\boxed{E}$ 230.

C

$-1^2 + 2^2 - 3^2 + 4^2 - 5^2 + 6^2 \dots - 19^2 + 20^2$

$-(2k-1)^2 + (2k)^2 = -(\cancel{4k^2} - 4k + 1) + \cancel{4k^2} = 4k - 1 \quad k = 1, 2 \dots 10.$

$4\sum_{i=1}^{10} k - \sum_{i=1}^{10} 1 = 4 \cdot \frac{10 \cdot 11}{2} - 10$

$= 4 \cdot 55 - 10 = 220 - 10 = 210.$

(4) Let $f(n) = \sum_{i=1}^{n} i$ and $g(n) = 4^{\log_2 n}$. What is the *most accurate* order relationship between f and g?

$\boxed{A}$ $f \in o(g)$.

$\boxed{B}$ $f \in O(g)$.

$\boxed{C}$ $f \in \Theta(g)$.

$\boxed{D}$ $f \in \Omega(g)$.

$\boxed{E}$ $f \in \omega(g)$.

C

$f(n) = \frac{1}{2} n(n+1)$

$g(n) = 2^{2\log_2 n} = 2^{\log_2 n^2} = n^2$

$f(n) \in \Theta(n^2)$

(5) Let $f(n)$ be a function satisfying the recurrence $f(0) = 0$; $f(n) = f(n-1) + \sqrt{n}$. Which order relationship describes f.

$\boxed{A}$ $f \in \Theta(n)$.

$\boxed{B}$ $f \in \Theta(n \log n)$.

$\boxed{C}$ $f \in \Theta(n\sqrt{n})$.

$\boxed{D}$ $f \in \Theta(n^2)$.

$\boxed{E}$ $f \in \Theta(n^3)$.

C

$f(n) = f(n-1) + \sqrt{n}$

$f(n-1) = f(n-2) + \sqrt{n-1}$

$\vdots$

$f(1) = f(0) + \sqrt{1}$

$f(n) = f(0) + \sqrt{1} + \dots \sqrt{n}$

$= \sum_{i=1}^{n} \sqrt{i}$

$\in \Theta(n^{3/2})$

(6) A class with 10 students needs to choose a president, vice-president and secretary (a student *cannot* fill multiple roles). In how many ways can this be done?

A 1000.

B 720. $10 \cdot 9 \cdot 8 = 720.$

B

C 120.

D 10!

E $\binom{10}{3}$.

(7) A fraternity orders 5 pizzas (eg. 2 with sausage and 3 with meatballs & onion). There are 5 toppings. A pizza can have 0,1 or 2 toppings. How many ways are there for the fraternity to make its order?

A 16. # different pizza's $= \binom{5}{0}+\binom{5}{1}+\binom{5}{2} = 1+5+10 = 16.$

B 16^5. ordering 5 pizzas: $\binom{16+5-1}{16-1} = \binom{20}{15}$

C $\binom{16}{5}$.

D. D $\binom{20}{15}$.

E $16 \times 15 \times 14 \times 13 \times 12$.

(8) A friendship network has 6 people Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ. If you add up the number of friends of each person, you get a total of 26. How many *different* social network graphs could correspond to this friendship network. (Two graphs are different if they don't have exactly the same edges.)

A 0. # of edges = 13 # possible edges $= \binom{6}{2} = \frac{6 \cdot 5}{2} = 15$

B 95. $\therefore$ # possible graphs $= \binom{15}{13} = \frac{15 \cdot 14}{2} = 15 \cdot 7 = \underline{105}$

C C 105.

D 115.

E 125.

(9) You are thinking of a graph with 5 nodes Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ. *Approximately* how many such graphs are there?

A 100.

B 500. # possible edges is $\binom{5}{2} = \frac{5 \cdot 4}{2} = 10$

C C 1000. # graphs $= 2^{10}$ [each edge could be there or not]

D 5000. $= 1024$

E 10000.

(10) X and Y are random variables (not necessarily independent). Which of the following is an expression for $Var(X+Y)$ (variance of the sum)?

A $Var(X) + Var(Y)$. $Var(X+Y) = E[(X+Y)^2] - E[X+Y]^2$

B $\mathsf{E}[(X+Y)^2]$. $= E[X^2] + E[Y^2] + 2E[XY] - E[X]^2 - E[Y]^2 - 2E[X]E[Y]$

C $\mathsf{E}[X^2] + \mathsf{E}[Y^2] - \mathsf{E}[X]^2 - \mathsf{E}[Y]^2$. $= E[X^2] - E[X]^2 + E[Y^2] - E[Y]^2 + 2E[XY] - 2E[X]E[Y]$

D. D $Var(X) + Var(Y) + 2\,\mathsf{E}[XY] - 2\,\mathsf{E}[X]\,\mathsf{E}[Y]$. $= \underline{Var(X) + Var(Y) + 2E[XY] - 2E[X]E[Y]}$.

E $Var(X) + Var(Y) - 2Var(XY)$.

(11) You independently generate two random ten bit binary sequences and compute a new sequence using the `bitwise-or` of the two random sequences (treating 0 as `false` and 1 as `true`). Let X be the number of 1s in the result. What is $\mathsf{E}[X]$. (for example, 0001110010 `bitwise-or` 1000111000 = 1001111010.)

[A] 2.5

[B] 3.5

[C] 5

[D] 6.5

[E] 7.5

E

$P[\text{bit wise or} = 1] = \frac{3}{4}$ ∴ $E[\sum b_i] = 10 \cdot \frac{3}{4} = 7.5$

(12) About 1 in a 1000 people have Coeliac disease. The outcome of a test for Coeliac is random: the test makes a mistake on 1 in 10 people who have it (90% accuracy if you have Coeliac); the test makes a mistake on 1 in 100 people who do not have it (99% accuracy if you do not have Coeliac). You got tested, and the result was positive. *Approximately* what are the chances that you have Coeliac?

[A] 0.1%

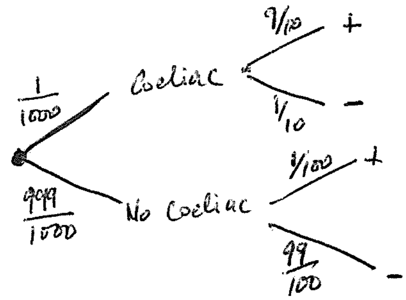
[B] 10%

[C] 40%

[D] 80%

[E] 90%

B



$\frac{1}{11}$

$\frac{\frac{9}{10} \cdot \frac{1}{1000}}{\frac{9}{10} \cdot \frac{1}{1000} + \frac{999}{1000} \cdot \frac{1}{100}} = \frac{\frac{9}{10000}}{\frac{9}{10000} + \frac{999}{100000}} = \frac{9}{9 + 99.9} \approx \frac{9}{108.9} \leq \frac{10}{100} = 10\%$

(13) Which set is *not countable*?

[A] {1,3,5,7}. ✓

[B] The prime numbers {2,3,5,7,. . . }. ✓

[C] All possible angles between 0 and 360.

[D] All even numbers which are not a sum of two primes. ✓

[E] All possible pairs of integers, $\mathbb{Z}^2$. ✓

C

(14) A random binary string $b_1 b_2 \ldots b_{10}$ of length 10 is the input to the automaton.
What is the probability that the string is accepted?

[A] 0.25

[B] 0.4

[C] 0.5

[D] 0.6

[E] 0.75

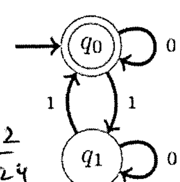
C.

accepted if # 1's is even.

even # 1's : $\binom{10}{0} + \binom{10}{2} + \binom{10}{4} + \binom{10}{6} + \binom{10}{8} + \binom{10}{10}$

$= \frac{1 + 45 + 210 + 210 + 45 + 1}{1024}$ 2^{10} $= \frac{512}{1024} = \frac{1}{2}$

Symmetry. Even # 1's should be as likely as odd.



(15) Which string below is *not* in the language of the CFG: $S \longrightarrow \varepsilon \mid 0S \mid S0 \mid 11S$

[A] ε ✓

[B] 1111 $S \to 11S \to 1111S \to 1111$ ✓

[C] 11011 $S \to 11S \to 110S \to 11011S \to 11011$ ✓

[D] 0011000 $S \to 0S \to 00S \to 0011S \to 00110S \to 001100S \to 0011000S \to 0011000$ ✓

[E] 001010 ← split 1's not possible

E

2 Positive Integer Partitions

A positive partition of n is a *sequence* of *positive* integers that add up to n. For example, $(6,4)$, $(4,6)$ and $(2,4,2,2)$ are different partitions of 10. How many positive partitions of n are there? Prove your answer.

Let $P(n)$ be the # of positive partitions of n.

$n=1$: 1 $\quad P(1)=1$

$n=2$: $(1,1)$ (2) $\quad P(2)=2$

$n=3$: $(1,1,1)$ $(1,2)$ $(2,1)$ (3) $\quad P(3)=4$

} looks like $P(n)=2^{n-1}$

Proof by induction (strong) that $P(n)=2^n$.

Base $\quad P(1)=1=2^{1-1}$ ✓.

Induction $\quad$ Assume $P(1)\wedge P(2)\wedge\cdots\wedge P(n)$.

Prove $P(n+1)$: # positive partitions of $n+1 = 2^n$.

A partition of n starts $(i, \ldots)$

$i=1,2,3,\ldots,n, n+1$

the remaining partition sums to $n-i$.

By the sum rule

$P(n+1) = \underbrace{\#\text{ starting } 1}_{\substack{\#\text{ positive}\\ \text{partitions } n-1\\ P(n+1)}} + \#\text{ starting } 2 + \cdots \#\text{ starting } n+1$

$\uparrow$ 1

$P(n+1)\ldots + P(1) + 1$

$\uparrow$ start with n.

$$= \underbrace{P(n) + P(n-1) + \cdots P(1)}\ +1$$

$$= \underbrace{2^{n-1} + 2^{n-2} + \cdots + 1}\ +1$$

$$= 2^n \cancel{-1} \cancel{+1}$$

$$= 2^n.$$ ✓ ▨

$\therefore$ By induction, $P(n)=2^{n-1}\ \forall n\geq 1$

3 Proofs

(a) *Prove* that $n^2 \leq 3^n$ for integer $n \geq 0$.

Proof by induction Base Case $0^2 = 0 \leq 3^0 = 1$. ✓.

Induction Assume $P(n): n^2 \leq 3^n$

Prove $P(n+1): (n+1)^2 \leq 3^{n+1}$

$$(n+1)^2 = n^2 + 2n + 1$$

$$\leq n^2 + n^2 + n^2 \quad \text{if } n \geq 2.$$

$$= 3n^2$$

$$\leq 3 \cdot 3^n = 3^{n+1}$$

$\therefore$ $P(n) \to P(n+1)$ for $n \geq 2$ $\boxtimes$

$P(2) \Rightarrow P(3) \to P(4) \dots$

$P(2)$ is T : $4 \leq 3^2 = 9$

$P(1)$ is T : $1 \leq 3$.

$P(0)$ is T : $0 \leq 1$

$\therefore$ $P(0), P(1), P(2)$, are true and $P(2)$ gives the claim

(b) *Prove* that $n^3 \notin O(n^2)$. You must *prove* that there is no *constant* C for which $n^3 \leq Cn^2$ for all $n \geq 1$.

Proof by contradiction Suppose $\exists\, C$ s.t. $n^3 \leq Cn^2$.

let $n \geq 2\lceil C \rceil$ [we can assume w.l.o.g $C \in \mathbb{N}$ otherwise we can take $\lceil C \rceil$

~~then~~

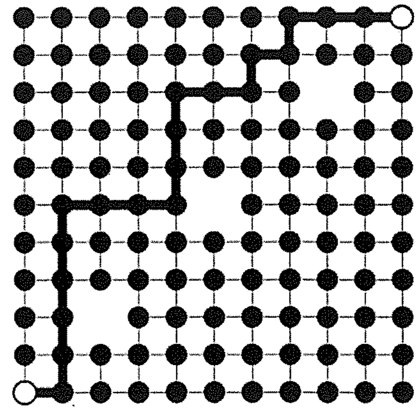
$$n^3 = 8\lceil C \rceil^3 \leq C \cdot n^2 = C \cdot \lceil C \rceil^2 \cdot 4$$

$$\therefore \quad 2\lceil C \rceil^3 \leq C \cdot \lceil C \rceil^2$$

$$\therefore \quad 2\lceil C \rceil \leq C.$$

which is a contradiction $\blacksquare$

4 Counting Paths on Graphs with Holes



A grid is missing nodes at $(2,2)$, $(5,5)$ and $(8,8)$. A *shortest* path from the bottom left node (0,0) to the top right node (10,10) is shown.

How many different shortest paths go from (0,0) to (10,10)? (Two paths are different if they do not have exactly the same edges).

You may leave your answer in the form of a combination of binomial coefficients – you do not need to compute a numerical answer.

Total # of paths from $(0,0)$ to $(10,10) = \binom{20}{10}$

paths going through $(2,2) = \binom{4}{2}\cdot\binom{16}{8}$

paths going through $(5,5) = \binom{10}{5}\cdot\binom{10}{5}$

paths going through $(8,8) = \binom{16}{8}\cdot\binom{4}{2}$

paths going through $(2,2)$ and $(5,5) = \binom{4}{2}\cdot\binom{6}{3}\cdot\binom{10}{5}$

$(5,5)$ and $(8,8) = \binom{10}{5}\cdot\binom{6}{3}\cdot\binom{4}{2}$

$(2,2)$ and $(8,8) = \binom{4}{2}\cdot\binom{12}{6}\cdot\binom{4}{2}$

paths going through $(2,2) + (5,5) + (8,8) = \binom{4}{2}\cdot\binom{6}{3}\cdot\binom{6}{3}\cdot\binom{4}{2}$

Use inclusion-exclusion to compute # paths using $(2,2)$ or $(5,5)$ or $(8,8)$:

A_1 = paths using $(2,2)$
A_2 = paths using $(5,5)$
A_3 = paths using $(8,8)$

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1\cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1\cap A_2 \cap A_3|$$

$$= \binom{4}{2}\cdot\binom{16}{8} + \binom{10}{5}^2 + \binom{16}{8}\cdot\binom{4}{2} - \binom{4}{2}\binom{6}{3}\binom{10}{5} - \binom{4}{2}^2\binom{12}{6} - \binom{4}{2}\cdot\binom{6}{3}\cdot\binom{10}{5} + \binom{4}{2}^2\binom{6}{3}^2$$

$$= 2\cdot\binom{4}{2}\binom{16}{8} + \binom{10}{5}^2 - 2\cdot\binom{4}{2}\binom{6}{3}\binom{10}{5} - \binom{4}{2}^2\binom{12}{6} + \binom{4}{2}^2\binom{6}{3}^2.$$

$\therefore$ # paths that do not use $(2,2)$ or $(5,5)$ or $(8,8) = \binom{20}{10} - |A_1 \cup A_2 \cup A_3|$

$$= \binom{20}{10} - 2\cdot\binom{4}{2}\binom{16}{8} - \binom{10}{5}^2 + 2\cdot\binom{4}{2}\binom{6}{3}\binom{10}{5} + \binom{4}{2}^2\binom{12}{6} - \binom{4}{2}^2\binom{6}{3}^2$$

← FULL CREDIT.

$= 46156$

5 Turing Machine and Exponentiation

(a) *Prove*: the problem (language) $\mathcal{L} = \{0^n\#1^{2^n} \mid n \geq 1\}$ *cannot* be solved (accepted) by a finite automaton.

Suppose it is accepted by a finite automaton with k-states. $q_0\, q_1\, q_2 \cdots q_{k-1}$

For input 0^{k+1} the automaton starts in $q_{i_0} = q_0$ and transitions to states $q_{i_1} q_{i_2} \cdots q_{i_k}$.

That is, the automaton visits $k+1$ states $\therefore$ by pigeonhole, two states visited are the same. So $q_{i_k} = q_{i_\ell}$ which means after processing

0^k and 0^ℓ, the automaton is in the same state

The automaton accepts $0^k\#1^{2^k}$ by construction

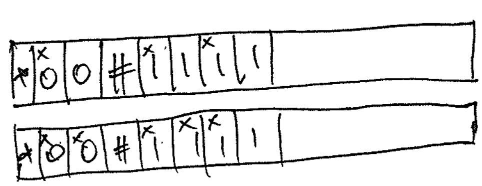
$\therefore$ the automaton also accepts $0^\ell\#1^{2^k}$ because after 0^ℓ the automaton is in the same state as after 0^k.

That is a contradiction because $0^\ell\#1^{2^k} \notin \mathcal{L}$, yet the automaton solves $\mathcal{L}$.

(b) Give a high-level description of a Turing Machine that solves $\mathcal{L} = \{0^n\#1^{2^n} \mid n \geq 1\}$.

The basic idea is to cross off half the uncrossed 1's for every zero.

1. Check the input has the right format.
2. Move right to the first unmarked 0 and mark it. (If no unmarked 0, check for one unmarked 1).
3. Move right to #. Check that there are an even # of UNMARKED 1's otherwise reject.
4. Move left to #
6. Mark EVERY OTHER UNMARKED 1.
7. Return to *



ACCEPT.