FINAL: 180 Minutes

Last Name: ______________________

First Name: ______________________

RIN: ______________________

Section: ______________________

Answer **ALL** questions. You may use **two** double sided $8\frac{1}{2} \times 11$ crib sheets.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

1	2	3	4	5	Total
200	**40**	**40**	**40**	**40**	**350**

(10 bonus points)

1 Circle at most one answer per question. 10 points for each correct answer.

(1) The **negation** of "Every student is a friend of some other student" is

- A Some student has a friend who is a student.
- B Some student is a friend of all students.
- C Some student is not a friend of some other student.
- D Some student is not a friend of all other students.
- E Some student has no friends.

(2) Estimate $2^1 \times 2^2 \times 2^3 \times \cdots \times 2^{20} = \prod_{i=1}^{20} 2^i$.

- A 1.65×10^{61}
- B 1.65×10^{63}
- C 1.65×10^{65}
- D 1.65×10^{67}
- E 1.65×10^{69}

(3) What is the *most accurate* order relation between 2^n and e^n?

- A $2^n \in o(e^n)$.
- B $2^n \in O(e^n)$.
- C $2^n \in \Theta(e^n)$.
- D $2^n \in \Omega(e^n)$.
- E $2^n \in \omega(e^n)$.

(4) $f(n)$ satisfies the recurrence $f(0) = 1$; $f(n) = nf(n-1)$. Which order relationship describes f.

- A $f \in \Theta(2^n)$.
- B $f \in O(2^n)$.
- C $f \in o(2^n)$.
- D $f \in \Theta(n^n)$.
- E $f \in o(n^n)$.

(5) What is the greatest common divisor of 756 and 840?

A 12.

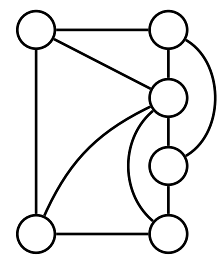
B 28.

C 63.

D 84.

E 189.

(6) What is the minimum number of colors needed to color the graph on the right?



A 2.

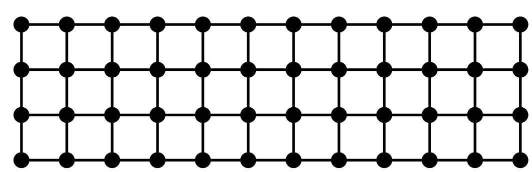
B 3.

C 4.

D 5.

E 6.

(7) On the right is the 4×12 grid graph. What is the average degree of a node?



A 3.

B $3\frac{1}{4}$.

C $3\frac{1}{3}$.

D $3\frac{1}{2}$.

E $3\frac{2}{3}$.

(8) Shirts come in 6 colors. 4 students are in a row. You must assign shirts to the students, and two students standing next to each other cannot get the same color shirt. In how many ways can you do this?

A $\binom{9}{3}$.

B $6 \times 5 \times 4 \times 3$.

C $\binom{6}{4}$.

D 6×5^3.

E 6^4.

(9) Pokemons have 4-digit serial numbers, e.g. 0255. A pokemon is defective if any digit repeats (e.g. 0255, 5250, 5255 are defective). *Approximately* what fraction of the possible serial numbers are defective?

[A] 0.

[B] 0.25.

[C] 0.5.

[D] 0.75.

[E] 1.

(10) A senate committee of 10 senators must pick a president. 3 candidates will be proposed from the 10 senators, and everyone votes. In how many ways can the 3 candidates be chosen.

[A] 1000.

[B] 720.

[C] 120.

[D] 10!

[E] $\frac{10!}{3!}$.

(11) Three integers z_1, z_2, z_3 satisfy $0 \leq z_1 \leq z_2 \leq z_3 \leq 6$ (the sequence is non-decreasing and bounded between 0 and 6). How many such sequences are there?

[A] 28.

[B] 42.

[C] 84.

[D] 165.

[E] 168.

(12) You are thinking of a graph with 4 nodes Ⓐ Ⓑ Ⓒ Ⓓ. How many such graphs are there?

[A] 24.

[B] 64.

[C] 81.

[D] 256.

[E] 4096.

(13) $\mathbf{X}, \mathbf{Y}$ are random variables (not necessarily independent) and $\mathbf{Z} = a\mathbf{X} + b\mathbf{Y}$. What is $\mathbb{E}[\mathbf{Z}]$?

[A] $a\,\mathbb{E}\,[\mathbf{X}] + b\,\mathbb{E}\,[\mathbf{Y}]$

[B] $a^2\,\mathbb{E}\,[\mathbf{X}] + b^2\,\mathbb{E}\,[\mathbf{Y}]$

[C] $(a+b)(\mathbb{E}[\mathbf{X}] + \mathbb{E}[\mathbf{Y}])$

[D] $a(\mathbb{E}[\mathbf{X}] + \mathbb{E}[\mathbf{Y}]) + b(\mathbb{E}[\mathbf{X}] + \mathbb{E}[\mathbf{Y}])$

[E] None of the above are true in general.

(14) This test has 20 multiple choice questions, each with 5 possible choices. If you answer questions randomly, what is the expected number of multiple questions you get correct?

[A] 3

[B] 4

[C] 5

[D] 6

[E] 10

(15) About 1 in a 1000 people have Coeliac disease. The test for Coeliac randomly makes a mistake 5% of the time (95% accuracy). You tested positive. *Approximately* what are the chances you have Coeliac?

[A] 0.2%

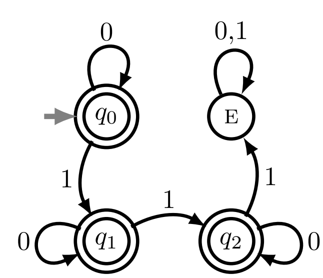
[B] 2%

[C] 20%

[D] 50%

[E] 95%

(16) A random binary string $b_1 b_2 \ldots b_{10}$ of 10 bits is the input to the automaton. What is the probability that the string is accepted?



[A] $\frac{2}{1024}$

[B] $\frac{45}{1024}$

[C] $\frac{56}{1024}$

[D] $\frac{90}{1024}$

[E] $\frac{512}{1024}$

(17) What is a computing problem?

A A Person.

B An automaton (machine which transitions between states as it reads the input).

C An automaton with stack memory.

D An automaton with random access memory.

E A set containing finite binary strings.

(18) The computing problem $\mathcal{L} = \{\text{strings with an even number of 1s}\}$ can be solved by:
(I) DFA. (II) CFG. (III) Turing Machine.

A I,II,III

B I,III

C II,III

D III only

E None of these models of computing

(19) The computing problem $\mathcal{L} = \{\text{strings corresponding to programs which HALT}\}$ can be solved by:
(I) DFA. (II) CFG. (III) Turing Machine.

A I,II,III

B I,III

C II,III

D III only

E None of these models of computing

(20) A DFA has *two* states a start state q_0 and a second state q_1. The DFA is described by a list of its accept states and a list of its transition instructions. The order in which you list the accept states and the transition instructions does not matter. We draw a DFA as a graph with nodes q_0, q_1 and add a directed arrow for each transition instruction (the accepting states have double circles).

How many different DFA's are there with two states? (*Different* DFA's *can* have the same YES-set)

A 4.

B 8.

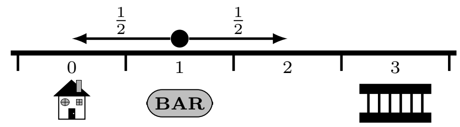
C 16.

D 32.

E 64.

2 Random Walk

A drunk leaves the bar (at position 1), and takes independent steps: left (L) with probability $\frac{1}{2}$ or right (R) with probability $\frac{1}{2}$. The drunk stops when he reaches home (at 0) or the jail (at 3). Compute the *expected* number of steps the drunk makes.



3 Induction

(a)
$$
\begin{aligned}
G(1) &= 1; \\
G(n) &= G(n-1)\left(1 - \frac{1}{n}\right) \quad \text{for } n > 1;
\end{aligned}
$$

Prove that $G(n) = \dfrac{1}{n}$ for integer $n \geq 1$.

(b) The nth Harmonic number is $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Prove that $H_1 + H_2 + \ldots + H_n = (n+1)H_n - n$.

4 Turing Machine

Give a high-level description of a Turing Machine that solves the problem $\mathcal{L} = \{0^n\texttt{\#}1^{n^2} \mid n \geq 0\}$ (squaring). (You may find it useful to illustrate how your TM works on 00#1111.)

5 [Hard] Unsolvable Problems

Prove: There is an undecidable computing problem which is a subset of $\{1\}^*$.

SCRATCH

SCRATCH