FINAL: 180 Minutes

Last Name: ____________________

First Name: ____________________

RIN: ____________________

Section: ____________________

Answer **ALL** questions. You may use **two** double sided $8\frac{1}{2} \times 11$ crib sheets.

You **MUST** show work (even for multiple choice) to receive full credit.

NO COLLABORATION or electronic devices. Any violations result in an F.

NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

1	2	3	4	5	Total
200	**40**	**40**	**40**	**40**	**350**

(10 bonus points)

1 Circle at most one answer per question. 10 points for each correct answer.

(1) Every card has a letter and a number. **Rule:** If a card has a P on it, then the other side *must* be a 5.

S **5** **P** **3**

Which of the above cards *must* be turned over to verify the rule has not been broken.

A **S** **5**

B **5** **P**

C **S** **3**

D **P** **3**

E None of the above.

(2) Which set relationship does not hold in general.

A $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

B $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

C $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

D $(A \cup B) \cap \overline{A} = B \cap \overline{A}$.

E They all hold in general.

(3) $T_0 = 2$ and $T_n = T_{n-1}^2$ for $n > 0$. Estimate T_{20}.

A $10^{3,156,500}$

B $10^{1,156,500}$

C $10^{315,650}$

D $10^{156,500}$

E $10^{31,565}$

(4) $T_0 = 2$ and $T_n = T_{n-1}^2$ for $n > 0$, as in problem (3). Which order relationship is accurate?

A $T_n \in O(n)$.

B $T_n \in O(2^n)$.

C $T_n \in O(n!)$.

D $T_n \in O(2^{n!})$.

E None of the above.

(5) What is the last digit of $3^{1000} \times 5^{2000} + 7^{3000} \times 9^{4000}$?

A 1.

B 2.

C 3.

D 4.

E None of the above.

(6) Let $d = \gcd(m, n)$, where $m, n > 0$. Bezout's identity gives $d = mx + ny$ where $x, y \in \mathbb{Z}$. Which of the statements A, B, C or D are false?

A It is always possible to choose $x > 0$.

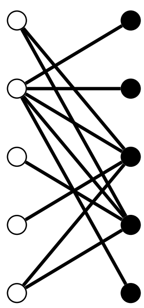
B It is always possible to choose $x < 0$.

C It is possible to find another $x, y \in \mathbb{Z}$ for which $0 < mx + ny < d$.

D It is always possible to find $a, b \in \mathbb{Z}$ for which $ax + by = 1$.

E All the statements A, B, C and D are true.

(7) The left nodes are tasks and the right nodes are resources. A resource can perform at most one task. What is the maximum number of tasks that can be performed?



A 1.

B 2.

C 3.

D 4.

E 5.

(8) A queen covers a square if that square is on the same row, column or diagonal as the queen. What is the minimum number of queens required to cover all squares on a 5×5 chessboard?

A 1.

B 2.

C 3.

D 4.

E 5.

(9) A friendship network has 100 people (vertices) and 2000 edges (friendships). You pick a person at random. What is the expected number of friends this person has?

- [A] 10.
- [B] 20.
- [C] 30.
- [D] 40.
- [E] None of the above, or not enough information to say for sure.

(10) To get into a certain US-college, all students submit at least one of SAT or ACT. 80% of students submit SAT; 40% of students submit ACT. How many students submit both SAT and ACT?

- [A] 10%.
- [B] 20%.
- [C] 30%.
- [D] 40%.
- [E] None of the above, or not enough information to say for sure.

(11) How many 4 digit strings (digits are 0,1,...,9) from 0000 to 9999 have digits which sum to 8. For example 0071, 0233 and 2033 are different digit-strings with digit-sum 8.

- [A] $\binom{8}{4} = 70$.
- [B] $\binom{11}{3} = 165$.
- [C] $10 \times 9 \times 8 \times 7 = 5040$.
- [D] $10^4 = 10,000$.
- [E] None of the above.

(12) How many different friendship networks are possible with the 5 people, Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ? (Two networks are different if they have different edge-sets.)

A Approximately 10.

B Approximately 100.

C Approximately 1000.

D Approximately 10,000.

E Approximately 100,000.

(13) A friendship network has 5 people, Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ. Each pair of people independently flips a fair coin and forms a friendship-edge if the flip is H. What is the probability that the network has exactly 5 edges?

A Approximately 1%.

B Approximately 2%.

C Approximately 10%.

D Approximately 25%.

E Approximately 50%.

(14) A friendship network has 5 people, Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ. Each pair of people independently flips a coin and forms a friendship if they get H. What is the expected number of edges in the friendship network?

A 2.

B 3.

C 4.

D 5.

E None of the above.

(15) A tennis club has 20 members who are paired up in twos for the first round of a tournament. In the first round, we only care about who plays whom. How many ways are there of forming the first round matches? *[Hint: With 4 members, there are 3 ways to form the first round matches.]*

A 20!.

B $\binom{20}{2}^{10}$.

C $\binom{20}{2} \times \binom{18}{2} \times \binom{16}{2} \times \binom{14}{2} \times \binom{12}{2} \times \binom{10}{2} \times \binom{8}{2} \times \binom{6}{2} \times \binom{4}{2} \times \binom{2}{2}$.

D $20!/(2^{10} \times 10!)$

E None of the above.

(16) $\mathbf{X}$ is a random variable and $\mathbf{Z} = a\mathbf{X} + b\mathbf{X}^2$. What is $\mathbb{E}[\mathbf{Z}]$?

A $\mathbb{E}[\mathbf{Z}] = a\,\mathbb{E}[\mathbf{X}] + b\,\mathbb{E}[\mathbf{X}]^2$

B $\mathbb{E}[\mathbf{Z}] = a\,\mathbb{E}[\mathbf{X}] + b^2\,\mathbb{E}[\mathbf{X}]^2$

C $\mathbb{E}[\mathbf{Z}] = a\,\mathbb{E}[\mathbf{X}] + b\,\mathbb{E}[\mathbf{X}^2]$

D $\mathbb{E}[\mathbf{Z}] = a\,\mathbb{E}[\mathbf{X}] + b^2\,\mathbb{E}[\mathbf{X}^2]$

E None of the above are true in general.

(17) $\mathbf{X}, \mathbf{Y}$ are *independent* random variables and $\mathbf{Z} = \mathbf{XY}$. What is $\sigma^2(\mathbf{Z})$, the variance of the product? *[Hint: Tinker with simple random variables. Make a conclusion and justify it.]*

A $\sigma^2(\mathbf{Z}) = \sigma^2(\mathbf{X})\sigma^2(\mathbf{Y})$

B $\sigma^2(\mathbf{Z}) = \sigma^2(\mathbf{X})\,\mathbb{E}[\mathbf{Y}^2] + \sigma^2(\mathbf{Y})\,\mathbb{E}[\mathbf{X}^2]$

C $\sigma^2(\mathbf{Z}) = \sigma^2(\mathbf{X})\,\mathbb{E}[\mathbf{Y}]^2 + \sigma^2(\mathbf{Y})\,\mathbb{E}[\mathbf{X}]^2$

D $\sigma^2(\mathbf{Z}) = \sigma^2(\mathbf{X})\,\mathbb{E}[\mathbf{Y}^2] + \sigma^2(\mathbf{Y})\,\mathbb{E}[\mathbf{X}]^2$

E None of the above are true in general.

(18) About 1 in a 100 people have Coeliac disease. The test for Coeliac has 90% accuracy, randomly making a mistake only 10% of the time. You tested positive. What are the chances you have Coeliac?

[A] 1/100.

[B] 1/12

[C] 1/8

[D] 1/4

[E] 9/10

(19) The computing problem $\mathcal{L} = \{0^{\bullet n}\, 1^{\bullet(n+m)}\, 0^{\bullet m} \mid m, n \geq 0\}$ can be solved by:

(I) DFA. (II) CFG. (III) Turing Machine.

[A] I,II,III

[B] I,III

[C] II,III

[D] III only

[E] None of these models of computing

(20) Which of these problems can be solved by a computer (Turing Machine)?

[A] Determine if some other program halts or loops forever – ULTIMATEDEGUGGER

[B] Determine (YES) or (NO) if some other program says (YES) on its input and halts.

[C] Given $n \in \mathbb{N}$, compute $f(n)$, where $f(n) = 1$ if the nth Turing Machine halts and 0 otherwise.

[D] Given m-bit and n-bit binary sequences $b_1 \cdots b_m$ and $c_1 \cdots c_n$ with $m < n$, is it possible to add $n - m$ bits into various positions of the first sequence so that the two sequences match exactly?

[E] None of these problems can be solved.

2 Independent Sets and Vertex Covers in a Graph. (Tinker, tinker,...)

A graph G has vertices $V = \{v_1, \ldots, v_n\}$ and edges $E = \{e_1, \ldots, e_m\}$. Let $S \subseteq V$ be a subset of the vertices.

S is a **vertex cover** if every edge in E has at least one endpoint in S.

S is an **inpdependent set** if no pair of vertices in S is connected by an edge.

Prove: The subset S is a vertex cover *if and only if* $\overline{S}$ (the vertices not in S) is an independent set.

3 Conditional Probability and Expected Value.

A box has 1 fair coin and 1 two-headed coin. You picked a random coin, flipped it 2 times and both flips were H. You now keep flipping the *same* coin you picked until you flip *two heads in a row.* Let **X** be the number of additional flips you make. Compute $\mathbb{E}[\mathbf{X}]$, the expected value of **X**.

4 Sums and Induction. (Tinker, tinker,...)

Obtain a formula that does not use a sum for $S(n) = \sum_{i=1}^{2n} (-1)^i i^2$. Prove your formula by *induction*.

5 Transducer Turing Machine for Unary to Binary.

Give a high-level description of a transducer Turing Machine to solve unary to binary conversion. The input is $0^{\bullet n}$ (if not reject). The Turing Machine should halt with the tape showing $0^{\bullet n}\#w$, where w is the binary representation of n. (E.g. for input 00000, the the tape should be 00000#101 when the machine halts.)

SCRATCH

SCRATCH