FINAL: 180 Minutes

Last Name: Solutions

First Name: ______

RIN: ______

Section: ______

Answer **ALL** questions. You may use **two** double sided $8\frac{1}{2} \times 11$ crib sheets.

You **MUST** show work (even for multiple choice) to receive full credit.

NO COLLABORATION or electronic devices. Any violations result in an F.

NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

1	2	3	4	5	Total
200	**40**	**40**	**40**	**40**	**350**

(10 bonus points)

1 Circle at most one answer per question. 10 points for each correct answer.

(1) Every card has a letter and a number. **Rule:** If a card has a P on it, then the other side *must* be a 5.

$\boxed{\text{S}}$ $\boxed{\text{5}}$ $\boxed{\text{P}}$ $\boxed{\text{3}}$

Which of the above cards *must* be turned over to verify the rule has not been broken.

A $\boxed{\text{S}}$ $\boxed{\text{5}}$

B $\boxed{\text{5}}$ $\boxed{\text{P}}$

C $\boxed{\text{S}}$ $\boxed{\text{3}}$

D $\boxed{\text{P}}$ $\boxed{\text{3}}$

E None of the above.

D

if $P \to 5$
false if LHS is T and RHS is F
∴ whenever RHS is F must check LHS is F → check $\boxed{3}$
whenever LHS is T must check RHS is T → check $\boxed{P}$

(2) Which statement is *not* true.

A $\overline{A \cap B} = \overline{A} \cup \overline{B}$. ✓

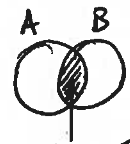
B $\overline{A \cup B} = \overline{A} \cap \overline{B}$. ✓

C $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. ✓

D $(A \cup B) \cap \overline{A} = B \cap \overline{A}$. ✓

E They are all true.

E



$A \cap B = \overline{\overline{A} \cup \overline{B}}$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$\overline{\overline{A} \cap \overline{B}} = \overline{\overline{A}} \cup \overline{\overline{B}} = A \cup B$

∴ $\overline{A} \cap \overline{B} = \overline{A \cup B}$

$\overline{A} \cap (A \cup B) = (\overline{A} \cap A) \cup (\overline{A} \cap B)$ $\to \phi$
(from part C)
$\to \overline{A} \cap (A \cup B) = \phi \cup \overline{A} \cap B$
$= \overline{A} \cap B$

(3) $T_1 = 2$ and $T_n = T_{n-1}^2$ for $n > 1$. Estimate T_{21}.

A $10^{3,156,500}$

B $10^{1,156,500}$

C $10^{315,650}$

D $10^{156,500}$

E $10^{31,565}$

C

$T_1 = 2^{2^0}$
$T_2 = 2^{2^1}$
$T_3 = (2^2)^2 = 2^{2 \times 2} = 2^{2^2}$
$T_4 = (2^{2 \times 2})^2 = 2^{2 \times 2 \times 2} = 2^{2^3}$
⋮
$T_{21} = 2^{2^{20}}$

$2^{20} \approx 10^6$
$2^{2^{20}} \approx 2^{10^6} \approx (2^{10})^{10^5} \approx (10^3)^{10^5}$
$= 10^{3 \times 10^5} = 10^{300\,000}$

(4) $T_1 = 2$ and $T_n = T_{n-1}^2$ for $n > 1$, as in problem (3). Which order relationship is accurate?

A $T_n \in O(n)$. ✗

B $T_n \in O(2^n)$. ✗

C $T_n \in O(n!)$. ✗

D $T_n \in O(2^{n!})$. ✓

E None of the above.

D

$T_n = 2^{2^{n-1}}$

$\frac{n}{2^{2^{n-1}}} \to 0$ $\quad \frac{2^n}{2^{2^{n-1}}} \to 0$ $\quad \frac{n!}{2^{2^{n-1}}} \to 0$ $\quad \frac{2^{n!}}{2^{2^{n-1}}} \to \infty$ because $\frac{n!}{2^{n-1}} \to \infty$.

(5) What is the last digit of $3^{1000} \times 5^{2000} + 7^{3000} \times 9^{4000}$?

E

- [A] 1.
- [B] 2.
- [C] 3.
- [D] 4.
- [E] None of the above.

$3^2 \equiv -1 \mod 10$

$\therefore (3^2)^{500} = 3^{1000} \equiv (-1)^{500} \equiv 1 \mod 10$

$5^2 \equiv 5 \mod 10 \to 5^k \equiv 5 \mod 10$

$3^{1000} \equiv 1 \mod 10 \to \underline{3^{1000} \times 5^{2000} \equiv 5 \mod 10}$

$5^{2000} \equiv 5 \mod 10$

$7^2 \equiv -1 \mod 10$. $9^2 \equiv 1$

$\to (7^2)^{1500} \equiv 1$ $(9^2)^{2000} \equiv 1$

$\therefore \underline{7^{3000} \times 9^{4000} \equiv 1}$

$3^{1000} \times 5^{2000} + 7^{3000} \times 9^{4000} \equiv 5 + 1 = 6.$

(6) Let $d = \gcd(m, n)$, where $m, n > 0$. Bezout's identity gives $d = mx + ny$ where $x, y \in \mathbb{Z}$. Which of the following are *not* true.

C

- [A] It is always possible to choose $x > 0$. ✓
- [B] It is always possible to choose $x < 0$. ✓
- [C] It is possible to find another $x, y \in \mathbb{Z}$ for which $0 < mx + ny < d$. ✗
- [D] It is always possible to find $a, b \in \mathbb{Z}$ for which $ax + by = 1$. ✓
- [E] All of the above are not true.

$d = mx + ny = m(x + \alpha n) + n(y - \alpha m)$

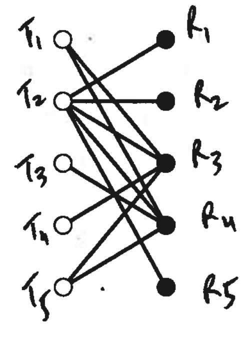
α is arbitrary

d is smallest possible

$\gcd(x, y) = 1$

otherwise can get a smaller d.

(7) The left nodes are tasks and the right nodes are resources. A resource can perform at most one task. What is the maximum number of tasks that can be performed?



C

- [A] 1.
- [B] 2.
- [C] 3.
- [D] 4.
- [E] 5.

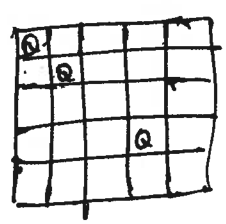
$N(T_1, T_3, T_4, T_5) = \{R_3, R_4\}$ $\therefore$ only two (at most) of $\{T_1, T_3, T_4, T_5\}$ can be done, eg. $T_1 - R_3$, $T_5 - R_4$

$\therefore$ can do at most $\underbrace{T_2 \text{ and two of } \{T_1, T_3, T_4, T_5\}}_{3.}$

(8) A queen covers a square if that square is on the same row, column or diagonal as the queen. What is the minimum number of queens required to cover all squares on a 5×5 chessboard?

C

- [A] 1.
- [B] 2.
- [C] 3.
- [D] 4.
- [E] 5.



3 is possible.

Can't do with 2.

(9) A friendship network has 100 people (vertices) and 2000 edges (friendships). You pick a person at random. What is the expected number of friends this person has?

$\boxed{A}$ 10.

$\boxed{B}$ 20.

$\boxed{C}$ 30.

$\boxed{D}$ 40.

$\boxed{E}$ None of the above, or not enough information to say for sure.

D

$E[\delta] = \sum \delta_i P_i$ $\quad \delta_i =$ degree

$P_i = \frac{1}{100}$

$= \frac{1}{100} \sum_i \delta_i$

$\sum_i \delta_i$ is sum of degrees $= 2|E| = 4000$

$= \frac{1}{100} \times 4000 = \underline{40}$

(10) To get into a certain US-college, all students submit either SAT or ACT scores. 80% of students submit SAT scores; 40% of students submit ACT scores. What percentage of students submitted both scores?

$\boxed{A}$ 10%.

$\boxed{B}$ 20%.

$\boxed{C}$ 30%.

$\boxed{D}$ 40%.

$\boxed{E}$ None of the above, or not enough information to say for sure.

B

$100\% = |SAT \cup ACT| = |SAT| + |ACT| - |SAT \cap ACT|$

$= 80\% + 40\% - \underline{|SAT \cap ACT|}$

$|SAT \cap ACT| = 80\% + 40\% - 100\%$

$= \underline{20\%}.$

(11) How many 4 digit strings (digits are 0,1,...,9) from 0000 to 9999 have digits which sum to 8. For example 0071, 0233 and 2033 are different digit-strings with digit-sum 8.

$\boxed{A}$ $\binom{8}{4} = 70$.

$\boxed{B}$ $\binom{11}{3} = 165$.

$\boxed{C}$ $10 \times 9 \times 8 \times 7 = 5040$.

$\boxed{D}$ $10^4 = 10,000$.

$\boxed{E}$ None of the above.

B

① Can do with the build-up counting method.

② Place 8 balls into 4 bins

$\equiv$ choose goody bag of size 8 from 4 candy colors.

$\binom{8+4-1}{4-1} = \binom{11}{3} = 165$

(12) How many different friendship networks are possible with the 5 people, Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ? (Two networks are different if they have different edge-sets.)

- [A] Approximately 10.
- [B] Approximately 100.
- [C] Approximately 1000.
- [D] Approximately 10,000.
- [E] Approximately 100,000.

C

$\#$ edges $= \binom{5}{2} = 10.$

Each edge has 2 choices (there or not)

$\therefore \#$ possibilities $= 2 \times 2 \times \ldots 2 = 2^{10} \simeq \underline{1000}.$

(13) A friendship network has 5 people, Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ. Each pair of people independently flips a fair coin and forms a friendship-edge if the flip is H. What is the probability that the network has exactly 5 edges?

- [A] Approximately 1%.
- [B] Approximately 2%.
- [C] Approximately 10%.
- [D] Approximately 25%.
- [E] Approximately 50%.

D

$P(\text{edge}) = 1/2$

$\#$ trials $n = 10$

$\therefore \; P(5 \text{ successes}) = \binom{10}{5} \frac{1}{2^{10}}.$

$= \frac{10!}{5! \times 5!} \times \frac{1}{2^{10}} = \frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1} \times \frac{1}{2^{10}}$

$= \frac{4 \times 9 \times 7}{2^{10}} = \frac{63 \times 4}{2^{10}} = \frac{252}{2^{10}} \simeq \frac{252}{1000} \simeq \underline{0.25}$

(14) A friendship network has 5 people, Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ. Each pair of people independently flips a coin and forms a friendship if they get H. What is the expected number of edges in the friendship network?

- [A] 2.
- [B] 3.
- [C] 4.
- [D] 5.
- [E] None of the above.

D

Expected $\#$ successes $= n \times p = 10 \times \frac{1}{2} = \underline{5}.$

edges.

(15) A tennis club has 20 members who are paired up in twos for the first round of a tournament. In the first round, we only care about who plays whom. How many ways are there of forming the first round matches? *[Hint: With 4 members, there are 3 ways to form the first round matches.]*

D

[A] 20!.

[B] $\binom{20}{2}^{10}$.

[C] $\binom{20}{2} \times \binom{18}{2} \times \binom{16}{2} \times \binom{14}{2} \times \binom{12}{2} \times \binom{10}{2} \times \binom{8}{2} \times \binom{6}{2} \times \binom{4}{2} \times \binom{2}{2}$.

[D] $20!/(2^{10} \times 10!)$

[E] None of the above.

Given first round matchings, we construct an ordering as follows:
① Construct ~~a random~~ an ordering of the pairs in 10! ways.
② Order within each pair in $2 \times 2 \times \cdots 2 = 2^{10}$ ways.

$\#$ first round matchings $\times 2^{10} \times 10! = 20!$

$\rightarrow$ $\#$ first round matchings $= \dfrac{20!}{2^{10} \times 10!}$

(16) $\mathbf{X}$ is a random variable and $\mathbf{Z} = a\mathbf{X} + b\mathbf{X}^2$. What is $\mathbb{E}[\mathbf{Z}]$?

C

[A] $\mathbb{E}[\mathbf{Z}] = a\,\mathbb{E}[\mathbf{X}] + b\,\mathbb{E}[\mathbf{X}]^2$

[B] $\mathbb{E}[\mathbf{Z}] = a\,\mathbb{E}[\mathbf{X}] + b^2\,\mathbb{E}[\mathbf{X}]^2$

[C] $\mathbb{E}[\mathbf{Z}] = a\,\mathbb{E}[\mathbf{X}] + b\,\mathbb{E}[\mathbf{X}^2]$

[D] $\mathbb{E}[\mathbf{Z}] = a\,\mathbb{E}[\mathbf{X}] + b^2\,\mathbb{E}[\mathbf{X}^2]$

[E] None of the above are true in general.

$$\underline{E[X] = a\,E[X] + b\,E[X^2]}$$

$\uparrow$ linearity of expectation

(17) $\mathbf{X}, \mathbf{Y}$ are *independent* random variables and $\mathbf{Z} = \mathbf{XY}$. What is $\sigma^2(\mathbf{Z})$, the variance of the product? *[Hint: Tinker with simple random variables. If you think one formula works, justify it.]*

D

[A] $\sigma^2(\mathbf{Z}) = \sigma^2(\mathbf{X})\sigma^2(\mathbf{Y})$

[B] $\sigma^2(\mathbf{Z}) = \sigma^2(\mathbf{X})\,\mathbb{E}[\mathbf{Y}^2] + \sigma^2(\mathbf{Y})\,\mathbb{E}[\mathbf{X}^2]$

[C] $\sigma^2(\mathbf{Z}) = \sigma^2(\mathbf{X})\,\mathbb{E}[\mathbf{Y}]^2 + \sigma^2(\mathbf{Y})\,\mathbb{E}[\mathbf{X}]^2$

[D] $\sigma^2(\mathbf{Z}) = \sigma^2(\mathbf{X})\,\mathbb{E}[\mathbf{Y}^2] + \sigma^2(\mathbf{Y})\,\mathbb{E}[\mathbf{X}]^2$

[E] None of the above are true in general.

$$\sigma^2(XY) = E[X^2Y^2] - E[XY]^2$$
$$= E[X^2]E[Y^2] - (E[X]E[Y])^2 \quad \leftarrow \text{independence.}$$
$$= E[X^2]E[Y^2] - E[X]^2E[Y]^2$$
$$= (\underbrace{E[X^2] - E[X]^2}_{\sigma^2(X)} + E[X]^2)\,E[Y^2] - E[X]^2E[Y]^2$$
$$= \sigma^2(X)E[Y^2] + E[X]^2E[Y^2] - E[X]^2E[Y]^2$$
$$= \sigma^2(X)E[Y^2] + E[X]^2(\underbrace{E[Y^2] - E[Y]^2}_{\sigma^2(Y)})$$
$$= \underline{\sigma^2(X)E[Y^2] + E[X]^2\sigma^2(Y)}$$

$\leftarrow$ looks asymmetric but its correct!

(18) About 1 in a 100 people have Coeliac disease. The test for Coeliac has 90% accuracy, randomly making a mistake only 10% of the time. You tested positive. What are the chances you have Coeliac?

[A] 1/100.

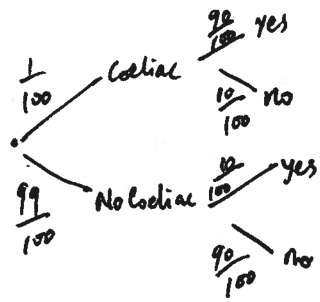
[B] 1/12

[C] 1/8

[D] 1/4

[E] 9/10

B



$P[\text{YES}] = \frac{90}{10000} + \frac{990}{10000}$

$P[\text{YES} \wedge \text{coeliac}] = \frac{90}{10000}$

$\therefore P(\text{Coeliac} \mid \text{YES}) = \dfrac{\frac{90}{10000}}{\frac{90+990}{10000}}$

$= \frac{9}{9+99} = \frac{1}{1+11} = \frac{1}{12}$

(19) The computing problem $\mathcal{L} = \{0^{\bullet n}\, 1^{\bullet(n+m)}\, 0^{\bullet m} \mid m, n \geq 0\}$ can be solved by:

(I) DFA. (II) CFG. (III) Turing Machine.

[A] I,II,III

[B] I,III

[C] II,III

[D] III only

[E] None of these models of computing

C

Use pigeonhole to show cannot be solved by DFA

$0^n 1^n$ is CFG. } concatenate → $0^n 1^{n+m} 0^m$
$1^m 0^m$ is CFG }

↑ CFG.

CFG → TM ∴ II and III

(20) Which of these problems can be solved by a computer (Turing Machine)?

[A] Determine if some other program halts or loops forever – ULTIMATEDEGUGGER ← ultimate debugger does not exist.

[B] Determine (YES) or (NO) if some other program says (YES) on its input and halts. ← L_{TM} undecidable

[C] Given $n \in \mathbb{N}$, compute $f(n)$, where $f(n) = 1$ if the nth Turing Machine halts and 0 otherwise. ← same as Ultimate debugger

[D] Given m and n-bit binary sequences $b_1 \cdots b_m$ and $c_1 \cdots c_n$ with $m < n$, is it possible to add $n - m$ bits into various positions of the first sequence so that the two sequences match exactly?

[E] None of these problems can be solved.

D

↑
Try all possible places to insert n-m bits. } slow but solvable.
If one works → accept
otherwise → reject

2 Independent Sets and Vertex Covers in a Graph

A graph G has vertices $V = \{v_1, \ldots, v_n\}$ and edges $E = \{e_1, \ldots, e_m\}$. Let $S \subseteq V$ be a subset of the vertices.

S is a **vertex cover** if every edge in E has at least one endpoint in S.

S is an **inpdependent set** if no pair of vertices in S is connected by an edge.

Prove: The subset S is a vertex cover *if and only if* $\overline{S}$ (the vertices not in S) is an independent set.

Two Parts to an if and only if

i) Suppose S is a Vertex cover.

Consider complement $\overline{S}$. If there is an edge between any pair in $\overline{S}$ then that edge does not have an endpoint in S $\therefore$ S is not a vertex cover

$\therefore$ No pair in $\overline{S}$ are connected by an edge $\rightarrow$ $\overline{S}$ is an independent set.

ii) Suppose $\overline{S}$ is an independent set.

Consider S. if some edge has no endpoint in S then both endpoints are in $\overline{S}$ which means $\overline{S}$ is not an independent set.

$\therefore$ every edge has at least 1 endpoint in $S \rightarrow S$ is a vertex cover

50% : understood problem and tinkered ~~[illegible]~~

80% : Proved two parts (started)

100% : Got both parts logic correct.

3 Conditional Probability and Expected Value

A box has 1 fair coin and 1 two-headed coin. You picked a random coin, flipped it 2 times and both flips were H. You now keep flipping the *same* coin you picked until you flip *two heads in a row.* Let **X** be the number of additional flips you make. Compute $\mathbb{E}[\mathbf{X}]$, the expected value of **X**.

$$P[\text{fair} \mid HH] = \frac{P[HH \mid \text{fair}]\, P[\text{fair}]}{P[HH]} = \frac{\frac{1}{4} \times \frac{1}{2}}{\frac{5}{8}} = \frac{\frac{1}{8}}{5/8} = \frac{1}{5}$$

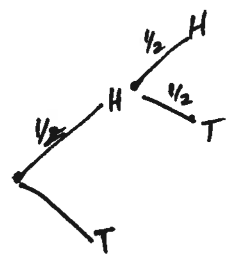
$$P[HH] = \frac{1}{4} \cdot \underbrace{P[\text{fair}]}_{\frac{1}{2}} + 1 \cdot \underbrace{P[\text{2-heads}]}_{\frac{1}{2}} = \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{2} = \frac{1}{8} + \frac{1}{2} = \frac{5}{8}.$$

Let $x =$ # flips till 2-heads in a row

$$E[x] = E[x \mid \text{fair}]\, P[\text{fair}] + \underbrace{E[x \mid \text{2-heads}]}_{2} P[\text{2-heads}]$$

$$= E[x \mid \text{fair}] \times \frac{1}{5} \qquad 2 \cdot \frac{4}{5}$$

$E[x \mid \text{fair}] =$



$$E[x] = \underbrace{E[x \mid HH]}_{2}\, \underbrace{P[HH]}_{\frac{1}{4}} + \underbrace{E[x \mid HT]}_{E[x]+2}\, \underbrace{P[HT]}_{\frac{1}{4}} + \underbrace{E[x \mid T]}_{E[x]+1}\, \underbrace{P[T]}_{\frac{1}{2}}.$$

$$= \frac{1}{2} + (E[x]+2)\frac{1}{4} + (E[x]+1)\frac{1}{2}$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + E[x]\underbrace{\left(\frac{1}{4} + \frac{1}{2}\right)}_{\frac{3}{4}}$$

$$\rightarrow \frac{1}{4} E[x] = \frac{3}{2}$$

$$\rightarrow E[x] = 6$$

$E[x \mid \text{fair}] = 6.$

$$\therefore E[x] = 6 \times \frac{1}{5} + 2 \times \frac{4}{5} = \frac{6+8}{5} = \frac{14}{5}$$

50%: Computed conditional probability of fair

80%: Understood to use total expectation

100%: Computed expectation.

4 Sums and Induction

Obtain a formula that does not use a sum for $S(n) = \sum_{i=1}^{2n} (-1)^i i^2$. Prove your formula by *induction*.

$$S(n) = \sum_{i=1}^{2n} (-1)^i (i)^2 = \sum_{j=1}^{n} (-1)^{2j-1}(2j-1)^2 + (-1)^{2j}(2j)^2.$$

$S(n) = \underbrace{-1+4}_{3}\underbrace{-9+16}_{7}\underbrace{-25+36}_{11}\cdots$

$$= \sum_{j=1}^{n} (2j)^2 - (2j-1)^2.$$

$$= \sum_{j=1}^{n} \cancel{(2j)^2} - (\cancel{(2j)^2} - 4j + 1)$$

$$= \sum_{j=1}^{n} 4j - 1 = 4\frac{n(n+1)}{2} - n = 2n(n+1) - n$$

$$= \underline{\underline{2n^2 + n}}.$$

$$S(n) = 2n^2 + n.$$

n	1	2	3
$S(n)$	3	10	21
$2n^2+n$	3✓	10✓	21✓

<u>Proof by induction</u>

$P(n)$: $S(n) = 2n^2 + n$.

<u>Base Case</u>. $P(1)$: $S(1) = 2 \times 1 + 1 = 3$ ✓.

<u>Induction</u> Assume $P(n)$: $S(n) = 2n^2 + n$.

Prove $P(n+1)$: $S(n+1) = \cancel{S(n)}\ 2(n+1)^2 + n+1$

$$S(n+1) = \sum_{i=1}^{2(n+1)} (-1)^i i^2 = \sum_{i=1}^{2n} (-1)^i i^2 + (-1)^{2n+1}(2n+1)^2 + (-1)^{2(n+1)}(2(n+1))^2.$$

$$= 2n^2 + n + 4(n+1)^2 - (4n^2 + 4n + 1)$$

$$= 2n^2 + n + 4(\cancel{n^2} + 2n + 1) - (\cancel{4n^2} + 4n + 1)$$

$$= 2n^2 + n + 4n + 3.$$

$$= 2n^2 + 4n + 2 + n + 1$$

$$= 2(n^2 + 2n + 1) + n + 1$$

$$= 2(n+1)^2 + (n+1) \quad ✓.$$

∎

50%: Tinker + understood problem

80%: Got formula.

100%: Proved by induction

5 Transducer Turing Machine for Unary to Binary

Give a high-level description of a transducer Turing Machine to solve unary to binary conversion. The input is $0^{\bullet n}$ (if not reject). The Turing Machine should halt with the tape showing $0^{\bullet n}\#w$, where w is the binary representation of n. (E.g. for input 00000, the the tape should be 00000#101 when the machine halts.)

Basic Idea.

Divide # zeros by 2.

$$n = 2k + b$$

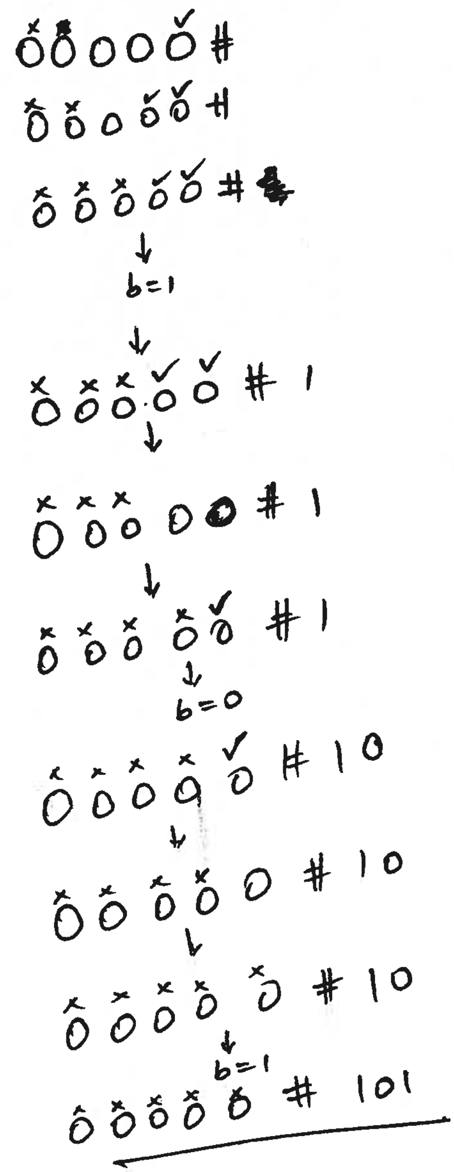
if $b=0$ → first bit is 0

if $b=1$ → first bit is 1

Now apply algorithm to K.

High Level TM

1. Check input for only zeros and add #
2. Mark zeros from left with x and corresponding zeros from right with ✓.
   if cannot mark ✓ → $b=1$
   → write b after # in ⊔
   otherwise → $b=0$
   → write b after # in ⊔
3. Erase all check-marks and repeat with all 0's between x and # mark.
4. If no more zeros, reverse the string after the #
5. At the end, erase all marks.



50% : Understood problem
80% : Understood Basic Idea
100% : Gave high-level code that is more-or-less correct.