FINAL: 180 Minutes

Last Name: Solutions

First Name: ____________

RIN: ____________

Section: ____________

Answer **ALL** questions. You may use **two** double sided $8\frac{1}{2} \times 11$ crib sheets.
You **MUST** show **CORRECT** work (even for multiple choice) to receive full credit.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

1	2	3	4	5	6	Total
200	**30**	**30**	**30**	**30**	**30**	**350**

1 Circle at most one answer per question. 10 points for each correct answer.

(1) Is this claim true or false. $\forall n \in \mathbb{Z} : n^2 \geq 0$.

[A] True. ← always non-negative

[B] False.

[C] You can't say because it depends on n.

[D] You can't assign true or false to quantified statements.

[E] It is not a proper statement to which you can assign true or false.

(2) If it rains on a day, it must rain the next day. Today it did not rain. What can you conclude?

[A] It won't rain tomorrow.

[B] It won't rain on any future day.

[C] It rained yesterday.

[D] It did not rain yesterday ~~but it could have rained on some day prior to yesterday~~.

[E] It did not rain yesterday and it did not rain on any day prior to yesterday.

Induction: If it rained on any previous day, it rains today $\therefore$

(3) To prove $P(n)$ by induction, which is *not* a valid induction step to prove $P(n) \to P(n+1)$.

[A] *Assume* that $P(n)$ is true and *prove* that $P(n+1)$ is true. ✓ T→F cannot occur

[B] *Assume* two things, that $P(n)$ is true and that $P(n+1)$ is false. Now derive a contradiction. ✓ T→F cannot occur.

[C] *Assume* that $P(n)$ is false and *prove* that $P(n+1)$ is false. ✗

[D] *Assume* that $P(n+1)$ is false and *prove* that $P(n)$ is false. ✓ Contrapositive

[E] All of the above are valid induction steps.

(4) What is the approximate value of the sum $\sum_{i=0}^{20}(2^i + i)(2^i - i)$.

No Work No Credit

[A] 1.5×10^{11}.

[B] 4.0×10^{11}.

[C] 1.5×10^{12}.

[D] 4.0×10^{12}.

[E] 1.5×10^{13}.

$$\sum_{i=0}^{20} 4^i - i^2 = \sum_{i=0}^{20} 4^i - \sum_{i=0}^{20} i^2$$

$i^2 \in o(4^i)$ $\therefore$ second term is small → ignore

$$\approx \sum_{i=0}^{20} 4^i = \frac{4^{21}-1}{4-1} \approx \frac{4^{21}}{3} = \frac{4\cdot(2^{20})^2}{3} \approx \frac{4\cdot(10^6)^2}{3} \approx \underline{1.33\cdot 10^{12}}$$

(5) $T_1 = 1$ and $T_n = T_{n-1} + n^2$ for $n > 1$. Which order relationship is accurate?

[A] $T_n \in \Theta(n)$.

[B] $T_n \in \Theta(n^2)$.

[C] $T_n \in \Theta(n^3)$.

[D] $T_n \in \Theta(2^n)$.

[E] None of the above.

$T_1 = 1$

$T_2 = 1 + 2^2$

$T_3 = 1 + 2^2 + 2^3$

$\vdots$

$T_n = \sum_{i=1}^{n} i^2 \in \Theta(n^3)$.

(6) What is the remainder when 2^{2019} is divided by 5?

$2^{2019} = 2 \cdot 2^{2018}$
$= 2 \cdot (2^2)^{1009}$

A 0.

$2^2 \equiv -1 \mod 5$

B 1.

No Work No Credit

C 2.

$\rightarrow (2^2)^{1009} \equiv (-1)^{1009} \equiv -1$

D 3.

E 4.

$2^{2019} = 2 \cdot (2^2)^{1009} \equiv -2 \equiv 3$

(7) Define the set $A = \{3x + 7y \mid x \text{ and } y \text{ are in } \mathbb{Z}\}$. Which numbers are *not* in A?

A -11.

$\gcd(3,7) = 1$

B 11.

$\therefore 1 = 3a + 7b$ Let $z \in \mathbb{Z}$

C 37.

$\rightarrow z = 3\underbrace{za}_{x} + 7\underbrace{zb}_{y}$ $\therefore A = \mathbb{Z}$.

D 142.

E They are all in A.

(8) Ayfos is in a social network with 14 others, so 15 people in all with Ayfos. There are 25 friendship links in this network. Everyone but Ayfos has 3 friends. How many friends does Ayfos have?

A 6.

$\underbrace{\delta_{Ayfos} + 14 \cdot 3}_{\text{sum of degrees}} = 2E = 2 \cdot 25 = 50$

B 7.

No work No Credit

C 8.

$\therefore \delta_{Ayfos} = 50 - 14 \cdot 3 = 8$

D 9.

E Can't be determined or such a social network cannot exist.

(9) In the previous problem regarding Ayfos' social network, you pick a person randomly. What is the expected number of friends that person has.

A $3\frac{1}{3}$.

$P[8] = \frac{1}{15}$ and $P[3] = \frac{14}{15}$

B $3\frac{1}{2}$.

No work No credit

$\therefore E[\text{degree}] = \frac{8}{15} + 3 \cdot \frac{14}{15} = \frac{50}{15} = 3\frac{1}{3}$.

C $3\frac{3}{4}$.

D 4.

E None of the above, or not enough information to say for sure.

(10) From 1000 students, 900 are CS and 200 are MATH. How many are CS-MATH duals?

A 50.

$1000 = |CS \cup Math| = |CS| + |Math| - |CS \cap Math|$

B 100.

$= 900 + 200 - |CS \cap Math|$

C 150.

D 200.

$\rightarrow |CS \cap Math| = 1100 - 1000$

E None of the above, or not enough information to say for sure.

$= 100.$

(11) Digits are 0,1,..., 9. How many of the three digit strings 000 to 999 have a digit-sum 10? (For example, 307 and 811 have digit sum 10, but 846 and 213 do not.)

Let $W(n,k)$ = # of k-digit strings summing to 10.

$W(n,k) = \sum_{i=0}^{\min(n,9)} W(n-i, k-1)$.

$W(n,1) = \begin{cases} 1 & n \le 9 \\ 0 & n \ge 10 \end{cases}$

We want $W(10,3)$

k \ n	0	1	2	3	4	5	6	7	8	9	10
3	1	3	6	10	15	21	28	36	45	55	63
2	1	2	3	4	5	6	7	8	9	10	9
1	1	1	1	1	1	1	1	1	1	1	0

sum of shaded

No Work No Credit

A 60.

B 63.

C 66.

D 69.

E None of the above.

(12) A and B are sets. $|A| = 5$ and $|B| = 3$. How many functions are there from A to B?

A 3^5.

B 5^3.

C $5!$.

D $\binom{5}{3}$.

E None of the above.

Each element of $A \to$ 3 options

$\therefore 3^5$

(13) A and B are sets. $|A| = 5$ and $|B| = 3$. How many injections (1-to-1) are there from A to B?

A 0.

B 100.

C 150.

D 200.

E None of the above.

$A \xrightarrow{INJ} B \to |A| \le |B|$

here $|A| > |B|$ so no injections

$A = \{a_1 a_2 a_3 a_4 a_5\}$ $B = \{b_1 b_2 b_3\}$

(14) A and B are sets. $|A| = 5$ and $|B| = 3$. How many surjections (onto) are there from A to B?

A 0.

B 100.

C 150.

D 200.

E None of the above.

No Work No Credit

Cases $\binom{5}{1}\cdot\binom{4}{3} + \binom{5}{2}\cdot\binom{3}{2} + \binom{5}{3}\cdot\binom{2}{1} = 5 \cdot 14 + 10 \cdot 6 + 10 \cdot 2$

$= 70 + 60 + 20 = \underline{150}$

$1 + \binom{4}{2} + \binom{4}{1}$ $\binom{3}{1}$

# of A's mapping to b_1 # A's mapping to b_2.

Inclusion Exclusion X_i = Functions not using b_i $\therefore |X_1 \cup X_2 \cup X_3| = 3 \cdot 2^5 - 3$

$= 96 - 3 = 93$

$|X_i| = 2^5$ $|X_i \cap X_j| = 1$ $|X_i \cap X_j \cap X_k| = 0$ $\therefore$ surjections $= 3^5 - 93 = 3 \cdot 81 - 93$

$= 243 - 93 = \underline{150}$

(15) You roll a die 4 times. What is the probability to get (exactly) 2 sixes?

A $6/6^4$.

B $12/6^4$.

C $36/6^4$.

D $150/6^4$.

E None of the above.

No Work No Credit

Binomial. $\binom{4}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2 = 6 \cdot \frac{25}{6^4} = \frac{25}{6^3}$.

$\frac{150}{6^4}$.

(16) Al and Jo each independently pick 4 restaurants randomly from 10 restaurants $r_1, \dots, r_{10}$. They must eat at a restaurant that both picked. Compute the probability they can eat at (exactly) 2 restaurants.

- [A] 2/7
- [B] 3/7
- [C] 4/7
- [D] 5/7
- [E] None of the above

No Work No Credit

We may assume Al picks the first 4 restaurants $r_1 r_2 r_3 r_4$.

\# ways for Jo to pick 4 = $\binom{10}{4}$

\# ways Jo picks 2 in the first 4 = $\binom{4}{2}\cdot\binom{6}{2}$

$\therefore$ ~~# ways~~ $P[\text{exactly } 2] = \dfrac{\binom{4}{2}\binom{6}{2}}{\binom{10}{4}} = \dfrac{6\cdot 15}{10\times 9\times 8\times 7}\cdot 4\times 3\times 2 = \dfrac{30}{70} = \dfrac{3}{7}$

(17) Compute the expected number of restaurants Al and Jo from the previous problem can eat at.

- [A] 1.2.
- [B] 1.4
- [C] 1.6.
- [D] 1.8
- [E] None of the above

$X_i = 1$ if both Al + Jo pick restaurant r_i

$P[X_i = 1] = \frac{4}{10}\cdot\frac{4}{10} = \frac{16}{100}$.

\# common restaurants $= X_1 + X_2 + \dots + X_{10}$

$E[\text{Common}] = 10\cdot E[X_i] = 10\cdot\frac{16}{100} = \frac{16}{10} = 1.6$

(18) Which computing problem *cannot* be solved by a DFA?

- [A] Strings with an even number of 1s. ✓
- [B] Strings which have more 1s than 0s. ✗ ← needs memory to "count" 0's and 1's.
- [C] Strings whose number of 1s is a multiple of 3. ✓
- [D] Strings whose number of 1s is not a multiple of 3. ✓
- [E] Each problem is solvable using a DFA

(19) Which string cannot be generated by the CFG $S \to \varepsilon|0S|1S$?

- [A] $11111111110000000000 = 1^{\bullet 10}0^{\bullet 10}$.
- [B] $10101010101010101010 = (10)^{\bullet 10}$.
- [C] $00000000000000000000 = 0^{\bullet 20}$.
- [D] $00110011001100110011 = (0011)^{\bullet 5}$.
- [E] They can all be generated.

$\varepsilon \to 0S \to 0,\ 00S,\ 01S$

$\to 1S \to 1,\ 10,\ 11S$

← All possible strings.

(20) Which answer is a valid conclusion about the decidability of the language $\mathcal{L}_B$?

- [A] $\mathcal{L}_A$ is decidable. A decider for $\mathcal{L}_B$ can be converted to a decider for $\mathcal{L}_A$. So, $\mathcal{L}_B$ is decidable.
- [B] $\mathcal{L}_A$ is decidable. A decider for $\mathcal{L}_A$ can be converted to a decider for $\mathcal{L}_B$. So, $\mathcal{L}_B$ is decidable.
- [C] $\mathcal{L}_A$ is undecidable. A decider for $\mathcal{L}_A$ can be converted to a decider for $\mathcal{L}_B$. So, $\mathcal{L}_B$ is undecidable.
- [D] $\mathcal{L}_A$ is undecidable. A decider for $\mathcal{L}_B$ can be converted to a decider for $\mathcal{L}_A$. So, $\mathcal{L}_B$ is decidable.
- [E] None of the above is valid.

2 Determine the Type of Proof and Prove

Prove that for $n \in \mathbb{N}$, $\sqrt{n(n+1)} \leq n + \frac{1}{2}$.

$\boxed{\text{Contradiction}}$ Assume $\sqrt{n(n+1)} > n + 1/2$

$\rightarrow \quad n(n+1) > (n+\frac{1}{2})^2 = n^2 + n + \frac{1}{4}$

$\rightarrow \quad n^2 + n > n^2 + n + 1/4$

$\rightarrow \quad 0 > 1/4$ CONTRADICTION ∎

So false.

$\boxed{\text{Proof By Induction}}$

$P(n): \sqrt{n(n+1)} \leq n + \frac{1}{2}$

Base $P(1): \sqrt{1\cdot(1+1)} \leq \frac{3}{2}$ or. $\sqrt{2} \leq \frac{3}{2}$ is TRUE.

Induction: Assume $\sqrt{n(n+1)} \leq n + \frac{1}{2}$.
Prove $\sqrt{(n+1)(n+2)} \leq (n+1) + \frac{1}{2}$.

$$(n+1)(n+2) = n^2 + 3n + 2$$
$$= n^2 + n + 2n + 2$$
$$= n(n+1) + 2(n+1)$$
$$\leq (n+\tfrac{1}{2})^2 + 2(n+1) \quad \text{(Induction hypothesis)}$$
$$= n^2 + n + \tfrac{1}{4} + 2(n+1)$$
$$= (n+1)^2 + (n+1) + \tfrac{1}{4}$$
$$= (n+1+\tfrac{1}{2})^2$$

$\rightarrow \quad \sqrt{(n+1)(n+2)} \leq n+1+\frac{1}{2}$ as was to be proved.

$\therefore$ By induction, $P(n)$ is true for all $n \geq 1$. ∎

STRICT GRADING
50%: Used one of the proof techniques + understood problem
80%: Made some progress with proof.
If using induction and did $P(n+1) \rightarrow P(n)$ No more than 50%
100%: Basically correct proof.

3 Induction and Sums. Tinker, Tinker, Tinker.

For $n \in \mathbb{N}$, obtain a formula for the sum $S(n) = \sum_{i=1}^{2n} (-1)^i i$ and prove your formula by induction.

$S(1) = -1 + 2 = 1$

$S(2) = \underbrace{-1+2}_{1} \underbrace{-3+4}_{1} = 2$

$S(3) = -1+2-3+4-5+6 = 3$

Looks like $S(n) = n$.

Proof By induction

Base $S(1) = 1$ ✓.

Induction

Assume $S(n) = n$.

Prove $S(n+1) = n+1$

$$S(n+1) = \sum_{i=1}^{2n+2} (-1)^i i = \sum_{i=1}^{2n} (-1)^i i + (-1)^{2n+1}(2n+1) + (-1)^{2n+2}(2n+2)$$

$$= n \underbrace{- (2n+1) + (2n+2)}_{1} \quad \text{(Induction Hypothesis)}$$

$$= n+1$$

As was to be proved

$\therefore S(n) = n \quad \forall n \geq 1.$ $\boxed{}$

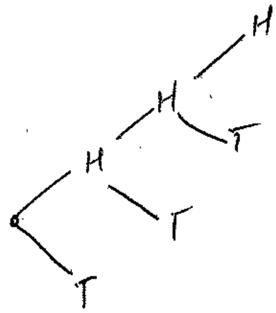
50%: Tinkered and came up with guess.

80%: Launched correct induction framework

100%: Basically correct proof.

4 Expected Waiting Time to 3 Heads In A Row

You flip a fair coin until you get 3 heads *in a row*. Compute the expected number of flips you make.



← Success.

Three ways the tossing starts

$\underbrace{T \quad HT \quad HHT}_{\text{RESTART}} \quad HHH.$

Let X = # flips

$$E[X] = \underbrace{E[X|T]}_{(E[X]+1)}\underbrace{P[T]}_{\cdot\frac{1}{2}} + \underbrace{E[X|HT]}_{(E[X]+2)}\underbrace{P[HT]}_{\cdot\frac{1}{4}} + \underbrace{E[X|HHT]}_{(E[X]+3)}\underbrace{P[HHT]}_{\cdot\frac{1}{8}} + \underbrace{E[X|HHH]P[HHH]}_{3\cdot\frac{1}{8}}$$

$$E[X] = \frac{E[X]}{2} + \frac{1}{2} + \frac{E[X]}{4} + \frac{1}{2} + \frac{E[X]}{8} + \frac{3}{8} + \frac{3}{8}.$$

$$= \frac{7E[X]}{8} + \frac{4+4+3+3}{8} \qquad \leftarrow \text{Solve for } E[X]$$

$$E[X] = 4+4+3+3$$
$$= 14$$

$$\boxed{\text{Expected \# of flips} = 14.}$$

50% Tinkered and showed understanding of problem

80% Used total probability and got it somewhat correct or used reasonable sum of infinite series

100% Basically correct. Algebra mistakes forgiven.

5 CFGs and Induction. (Tinker, tinker,...)

For the CFG $S \to 0|0S1$, *prove* that every string that can be generated has odd length.

Induction on the length of the derivation

$\underline{n=1}$ $S \to 0$ length $= 1$ which is odd.

Assume $P(n)$: Any length n derivation has odd length.

Prove $P(n+1)$: Any length $n+1$ derivation has odd length.

Such a derivation starts

$$S \Rightarrow 0S1 \xRightarrow{*} 0x1.$$

Where $S \overset{*}{\Rightarrow} x$ in a length n derivation

$\therefore$ by the induction hypothesis, $|x|$ is odd

$\therefore$ $|0x1| = 2 + |x|$ which is odd, as was to be proved.

$\therefore$ by induction, $P(n)$ is true for $n \geq 1$.

That is any derivation produces a string of odd length ∎

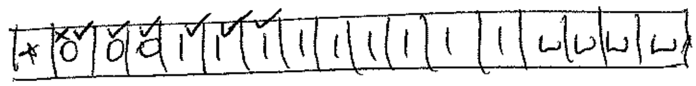
50%: Tinkered and showed understanding of problem

80%: set up induction correctly.

100%: Correct proof.

6 Turing Machine for Squaring.

Give a high level pseudo-code description of a Turing Machine that solves the problem $\mathcal{L} = \{0^{\bullet n}1^{\bullet n \times n} | n \geq 1\}$. (You do not need to give machine level details but your pseudo-code should demonstrate understanding of how the Turing Machine moves back and forth to solve the problem. Tinker.)



Basic Idea: For each unmarked 0, mark a number of 1's equal to the # of 0's.

① Check input is 0s followed by 1's.

② Mark a 0 with a X. If no more unchecked (X) 0, GOTO ⑤

③ For each non-✓ marked 0, mark with ✓ and mark a 1 with ✓.
if no 1's to mark → REJECT.

④ Go back to * removing ✓ from all 0's and GO TO ②.

⑤ Check if unmarked 1's remain. If so, REJECT
If not ACCEPT.

50%: Tinkered and showed correct example of how basic idea works

80%: Reasonable attempt at pseudocode.

100%: Correct high level description of TM.