FINAL: 180 Minutes

Last Name: _____

First Name: _____

RIN: _____

Section: _____

Answer **ALL** questions. You may use **two** double sided $8\frac{1}{2} \times 11$ crib sheets.

You **MUST** show **CORRECT** work (even for multiple choice) to receive full credit.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

1	2	3	4	5	6	Total
200	**30**	**30**	**30**	**30**	**30**	**350**

INSTRUCTIONS

1. This is a **closed book** test. No electronics, books, notes, internet, etc.
2. You can have two double sided $8\frac{1}{2} \times 11$ crib-sheets (handed in separately).
3. The test will become available in Submitty at 8am on the test date. Your PDF is due in Submitty by 8am the next day. You have 3 hours to do the exam and 3 additional hours to type your answers and submit a PDF.
4. By submitting the test you attest that the work is entirely your own and you obeyed the time limits of the exam.
5. Your submission *must* be typed PDF. The 3 hour test time for solving the problems must be continuous. The extra 3 hours is to type your answers and explanations: you may take breaks, but not change answers.
6. You *must* show your work for *every* answer immediately after the answer. The format for what you hand in is something like:

> **Problem 1**
> (1) A
> Because x is even, therefore ...
> (2) B
> Because $\sqrt{2}$ is irrational, therefore ...
> (4) D
> By the law of total expectation, $\mathbb{E}[\mathbf{X}] = \cdots$
> ⋮
> (20) A
> We proved in class that $\ell = n - 1$. Therefore ...
> **Problem 2**
> ⋮

   – Start each long-answer question on a new page.
   – Some problems may be "easy", so give a short explanation.
   – Some problems may require a detailed reasoning.
   – `3*3+1+3=13` is **not** an explanation. Everyone knows that `3*3+1+3=13`.
   Why this equation? Where do the numbers come from?
7. **If you don't show correct work, you won't get credit.**
8. Be especially careful in the multiple choice.
   – Correct answers get 10 points.
   – Wrong answers or correct answers without correct justification get 0.
9. Submit with plenty of time to spare. A late test won't be accepted.

1 Circle at most one answer per question. 10 points for each correct answer.

(1) "For a constant $c > 0$, $1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > c\sqrt{n}$, where n is any natural number." Which claim is this?

[A] $\exists c > 0 : (\exists n \in \mathbb{N} : 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > c\sqrt{n})$.

[B] $\exists c > 0 : (\forall n \in \mathbb{N} : 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > c\sqrt{n})$.

[C] $\exists n \in \mathbb{N} : (\forall c > 0 : 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > c\sqrt{n})$.

[D] $\forall n \in \mathbb{N} : (\exists c > 0 : 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > c\sqrt{n})$.

[E] None of the above.

(2) You will pick a constant $C > 0$ such that no matter which $n \in \mathbb{N}$ I pick, $\sum_{i=1}^{n} i \leq Cn$. Which is true?

[A] You can pick a C satisfying $C \leq 10$.

[B] You can pick a C satisfying $10 < C < 100$.

[C] You can pick a C satisfying $100 < C < 1000$.

[D] You can pick a C satisfying $1000 < C$.

[E] There is no constant $C > 0$ that you can pick.

(3) $T_1 = 2$ and $T_n = T_{n-1} + 2n$ for $n > 1$. What is T_{100}?

[A] 5050.

[B] 10100.

[C] 20200.

[D] 40400.

[E] None of the above.

(4) $T_1 = 1$ and $T_n = n \times T_{n-1}$ for $n > 1$. Which is true?

[A] $T(n) \in O(n^2)$.

[B] $T(n) \in o(2^n)$.

[C] $T(n) \in \Theta(2^n)$.

[D] $T(n) \in \omega(2^n)$.

[E] None of the above.

(5) You divide 2^{2016} candies evenly among 11 kids. How many candies are left over?

[A] 0.

[B] 3.

[C] 6.

[D] 9.

[E] None of the above.

(6) Estimate the sum $S = \sum_{i=1}^{\infty} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$.

A $0 < S \leq 2$.

B $2 < S \leq 2000$.

C $2000 < S \leq 20000$.

D $20000 < S \leq 200000$.

E None of the above.

(7) How many of the numbers $100, 101, 102, \ldots, 999$ do not contain the digit 2?

A 100.

B 504.

C 648.

D 729.

E None of the above.

(8) Let S be the sum of the reciprocals of all natural numbers not containing the digit 2. Estimate S.

A $0 < S \leq 2$.

B $2 < S \leq 2000$.

C $2000 < S \leq 20000$.

D $20000 < S \leq 200000$.

E None of the above.

$$S = 1 + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} + \tfrac{1}{9} + \tfrac{1}{10} + \tfrac{1}{11} + \tfrac{1}{13} + \tfrac{1}{14} + \tfrac{1}{15} + \tfrac{1}{16} + \tfrac{1}{17} + \tfrac{1}{18} + \tfrac{1}{19} + \tfrac{1}{30} + \tfrac{1}{31} + \tfrac{1}{33} + \tfrac{1}{34} + \tfrac{1}{35} + \cdots$$

(9) Shirts come in 3 colors R, G or B. In how many ways can you distribute shirts to 7 students?

A $\binom{7}{3}$.

B 7^3.

C 3^7.

D $7!/3!$.

E None of the above.

(10) Repeat problem 9 if at least two shirts of each color are distributed to students (7 shirts in total).

A 570

B 600.

C 630.

D 660.

E None of the above.

(11) Every vertex in a graph G has degree 1. Which is true?

A The graph G must be disconnected.

B The graph G could have 5 vertices.

C The graph G must have a cycle.

D The graph G is not possible.

E None of the above

(12) You rolled a pair of dice. What are the chances you rolled exactly one 5?

A 9/36.

B 10/36.

C 11/36.

D 12/36.

E None of the above.

(13) You rolled a pair of dice. What are the chances you rolled exactly one 5 if the sum is even?

A 4/10.

B 5/10.

C 4/11.

D 5/11.

E None of the above.

(14) Which of the following random variables $\mathbf{X}$ is not a binomial random variable.

A Randomly throw 100 darts at a dart board. $\mathbf{X}$ is the number of darts hitting the bulls-eye.

B Randomly answer 100 5-choice multiple choice questions. $\mathbf{X}$ is the number of questions correct.

C Randomly answer 100 5-choice multiple choice questions. $\mathbf{X}$ is the number of questions wrong.

D 1000 students randomly line up, 500 are boys. $\mathbf{X}$ is the number of boys in the first 100 students.

E They are all binomial random variables.

(15) A social network (graph) is a *tree* with 20 people. The edges are friendships. Each person randomly picks red or blue. Friends compare to see if they match. What is the expected number of matches.

A 4.75.

B 5.

C 9.5

D 10.

E None of the above or not enough information.

(16) On BlueToe, your first child is equally likely to be a boy or girl. From then on, the sex of a child is the same as the previous child with probability 2/3 and different with probability 1/3. What is the expected number of kids to get a girl?

A 1.5.

B 2.

C 2.5.

D 3.

E None of the above.

(17) On BlueToe, as in problem 16, what is the expected number of kids to two girls?

A 3.25.

B 4.

C 4.5.

D 5.25.

E None of the above.

(18) Estimate the number of DFA you can draw with 4 states, q_0, q_1, q_2, q_3. Tinker!

A About a hundred.

B About a thousand.

C About a million.

D About a billion.

E About a trillion.

(19) Which string can be generated by the CFG $S \to 0|1|SSS$?

A 1111.

B 0000.

C 000111.

D 111000.

E None of the above.

(20) If $\mathcal{L}_A$ is decidable, then $\mathcal{L}_B$ is decidable. We know that $\mathcal{L}_B$ is undecidable. Therefore:

A $\mathcal{L}_A$ must be finite.

B $\mathcal{L}_A$ must be infinite.

C $|\mathcal{L}_A| > |\mathcal{L}_B|$.

D $|\mathcal{L}_B| < |\mathcal{L}_A|$.

E None of the above.

2 Determine the Type of Proof and Prove

Prove that there is a constant $c > 0$ for which, no matter which $n \in N$ you pick,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > c\sqrt{n}.$$

3 Product of 5 Consecutive Numbers.

Prove that the product of any 5 consecutive natural numbers is divisible by 5! (e.g. $5!|3 \times 4 \times 5 \times 6 \times 7$).

4 Expected Waiting Time to All Colors of Starbust.

Starburst is sold in 2-packs, and there are 3 colors of starbust. What is the expected number of 2-packs you will buy if your goal is to get all colors?

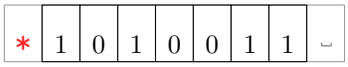
5 DFA or no DFA

Give a DFA for $\mathcal{L} = \{0^{\bullet^{n^2}} | n \geq 1\} = \{0, 0000, 000000000, \ldots\}$, or prove that $\mathcal{L}$ can't be solved with DFA.

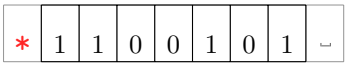
6 Transducer Turing Machine for Reversal.

Give a high level pseudo-code description of a transducer Turing Machine for reversal. The input on the tape is any binary string w. When the Turing Machine halts, the reversal of w should have replaced w. E.g:

Start



End



(Don't give machine level details, but you should make it clear how the Turing Machine moves back and forth. Tinker.)

SCRATCH

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