MIDTERM: 90 Minutes

Last Name: Solutions

First Name: ____________

RIN: ____________

Section: ____________

Answer **ALL** questions. You may use a single sided $8\frac{1}{2} \times 11$ crib sheet.

NO COLLABORATION or electronic devices. Any violations result in an F.

NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

1	2	3	4	5	Total
50	50	50	50	50	250

1 Circle at most one answer per question. 10 points for each correct answer and -5 points for each incorrect answer (blank answer is 0 points). Don't guess!

(a) $p \to (q \land r)$ is equivalent to what other compound proposition:

$\boxed{A}$ $(p \to q) \land r$

$\boxed{B}$ $(p \to q) \land (p \to r)$

$\boxed{C}$ $(p \land q) \to r$

$\boxed{D}$ $p \lor (q \land r)$

$\sim p \lor (q \land r) \equiv (\sim p \lor q) \land (\sim p \lor r)$

$\equiv (p \to q) \land (p \to r).$

B

(b) The negation of "If Malik is in pajamas then all lights are off" is

$\boxed{A}$ Malik is in pajamas and at least one light is on

$\boxed{B}$ Malik is in pajamas or all lights are off

$\boxed{C}$ Malik is not in pajamas and at least one light is on

$\boxed{D}$ Malik is not in pajamas and all lights are off

$\sim(p \to q) \equiv \sim(\sim p \lor q)$

$\equiv p \land \sim q.$

p: Malik is in pajamas

q: light are off

~q: at least one light is on

A

(c) $P(n)$ is a predicate (n is an integer). $P(2)$ is true; and, $P(n) \to (P(n^2) \land P(n-2))$ is true for $n \geq 2$. For which n can we be sure $P(n)$ is true?

$\boxed{A}$ All $n \geq 2$.

$\boxed{B}$ All even $n \geq 0$.

$\boxed{C}$ All odd $n \geq 0$.

$\boxed{D}$ All n which are perfect squares.

$P(0)$ $P(2) \to P(4) \longrightarrow P(16) \longrightarrow \ldots$

$\ldots P(14)$

B

(d) Compute the remainder when 2014^{2014} is divided by 5? *[Hint: $2014 \equiv -1 \pmod 5$.]*

$\boxed{A}$ $r = 1$

$\boxed{B}$ $r = 2$

$\boxed{C}$ $r = 3$

$\boxed{D}$ $r = 7$

$2014 \equiv -1 \mod 5$

$(2014)^{2014} \equiv (-1)^{2014} \mod 5$

$\uparrow 1.$

A

(e) A friendship network has 7 people and each person has 5 friends. How many edges (friendship links) are there in this friendship network?

$\boxed{A}$ 17 edges

$\boxed{B}$ 18 edges

$\boxed{C}$ Not enough information to determine the number of edges

$\boxed{D}$ This friendship network cannot possibly exist

$\sum d_i = 7 \cdot 5 = 2E$

$\Rightarrow 2E = 7 \cdot 5$

$\Rightarrow E = \frac{35}{2} = 17.5$

D.

2 Induction Proofs

1. Prove by induction that for all integer $n \geq 1$: $\sum_{i=1}^{n} \frac{1}{i(i+1)} = 1 - \frac{1}{n+1}$.

Base Case $P(1)$: $\frac{1}{1 \cdot 2} = 1 - \frac{1}{2}$ ✓

Induction: Assume $P(n)$: $\sum_{i=1}^{n} \frac{1}{i(i+1)} = 1 - \frac{1}{n+1}$

Prove $P(n+1)$: $\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = 1 - \frac{1}{n+2}$

$$\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \frac{1}{i(i+1)} + \frac{1}{(n+1)(n+2)} =$$

$$= 1 - \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} \quad [IH].$$

$$= 1 - \frac{1}{n+1}\left[1 - \frac{1}{n+2}\right]$$

$$= 1 - \frac{1}{\cancel{n+1}}\left[\frac{\cancel{n+2-1}}{n+2}\right]$$

$$= 1 - \frac{1}{n+2} \qquad \therefore P(n+1) \text{ is T} \quad \blacksquare$$

2. Suppose $a \equiv b \pmod{k}$. Prove by induction that for all integer $n \geq 1$: $a^n \equiv b^n \pmod{k}$.
   ($x \equiv y \pmod{z}$ means $x - y$ is divisible by z.)

Base Case $n=1$: $a^1 \equiv b \pmod{k}$ given

Induction Assume $a^n \equiv b^n \bmod k$.

Prove $a^{n+1} \equiv b^{n+1} \bmod k$.

$a^n - b^n$ is divisible by $k \rightarrow a^n - b^n = \alpha k$. also we know $a - b = \beta k$.

$\rightarrow a = b + \beta k$

$\therefore a^n = b^n + \alpha k$

$$a \cdot a^n = a \cdot (b^n + \alpha k) = (b + \beta k)(b^n + \alpha k)$$

$$= b^{n+1} + \beta k (b^n + \alpha k) + \alpha \beta k^2$$

$$\rightarrow a^{n+1} - b^{n+1} = \underbrace{k[\beta(b^n + \alpha k) + \alpha\beta k]}_{\text{a multiple of } k} \rightarrow k \mid a^{n+1} - b^{n+1}$$

$\rightarrow a^{n+1} \equiv b^{n+1} \pmod{k}$

$\blacksquare$

3 Well formed arithmetic expressions

Define a set $\mathcal{A}$ of well formed arithmetic strings (sequences) with alphabet $\Sigma = \{1, +, \times, (,)\}$.

[Base Case] $1 \in \mathcal{A}$;

[Recursive Rules] 1. $x, y, z \in \mathcal{A} \rightarrow (x + y + z) \in \mathcal{A}$
2. $x, y \in \mathcal{A} \rightarrow (x \times y) \in \mathcal{A}$.

must have outer
↓ parenthesis.

(a) Of the following three strings, circle the one that is in $\mathcal{A}$.

$(1 + 1 + 1) \times (1 + 1)$ $\quad$ $(1 + 1 + 1) \times ((1 + 1 + 1) + 1 + 1)$ $\quad$ $((1 + 1 + 1) \times ((1 + 1 + 1) + 1 + 1))$

(b) Give a derivation of the string in (a) that is in $\mathcal{A}$. (A derivation is a sequence of strings in $\mathcal{A}$ where each string is obtained from the previous strings by applying one of the recursive rules.)

$1 \xrightarrow{\text{rule 1}} (1+1+1)$

$\xrightarrow{\text{rule 1}} ((1+1+1)+1+1)$

$\xrightarrow{\text{rule 2}} ((1+1+1) \times ((1+1+1)+1+1))$.

4 Evaluating arithmetic expression strings

The function eval takes an input string from the set $\mathcal{A}$ of well formed arithmetic expressions ($\mathcal{A}$ was defined in problem 3) and computes its value as an arithmetic expression. For example,

$$\text{eval} : ((1+1+1) \times (((1+1+1) \times (1+1+1)) + 1 + 1)) \mapsto 33$$

Prove that for every string $x \in \mathcal{A}$, $\text{eval}(x)$ is odd.

Structural Induction   Base case   eval(1) = 1   odd ✓

Induction   2 Parts, one for each constructor.

① Show   eval(x), eval(y), eval(z) odd → eval((x+y+z)) odd

x+y+z is sum of 3 odd numbers

∴ odd ✓

② Show   eval(x), eval(y) odd → eval((x*y)) odd

x · x y = product of 2 odd numbers

∴ odd.

Since both recursive rules. preserve the property eval(s) is odd,

by structural induction eval(S) is odd for any $s \in \mathcal{A}$.

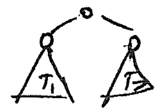
Is $((1+1+1) \times ((1+1+1)+1+1+1))$ in $\mathcal{A}$? If yes, give a derivation. If no, prove it.

evaluates to 3 × 6 = 18 (not odd) ∴ not in $\mathcal{A}$.

5 Rooted binary trees (NOT rooted full binary trees)

Give the recursive definition of rooted binary trees. Explicitly state your base case and recursive rules.

① $\varepsilon \in RBT$

②  $\in RBT \longrightarrow$  $\in RBT$

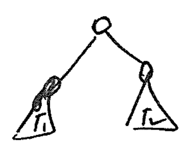
Let F be the number of full nodes (with 2 children) and L the number of leaf nodes (with no children). For any non-empty rooted binary tree, prove that

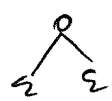
$$L = F + 1.$$

Claim if x is non empty RBT $\longrightarrow L = F+1$

Base case ε $\longleftarrow$ empty so claim is true.

Induction suppose. $L = F+1$ for 

when $T_1 T_2$ are not empty then: we must show  satisfies $L=F+1$

Case 1 T_1, T_2 is empty $\rightarrow$  $=$ o is created $L=1$ $F=0$

$L = F+1$

Case 2. T_1 is empty T_2 is not  is created.

assume L_2 and F_2 in T_2 with $L_2 = F_2 + 1$

in the tree created $L = 0 + L_2$ $F = F_2$ $\therefore L = F+1$

Similar if T_1 is not empty and T_2 is empty.

Case 3 T_1 is not empty T_2 is not empty:  $L = L_1 + L_2 = F_1 + 1 + F_2 + 1$

$F = F_1 + F_2 + 1$

$\therefore L = F+1$

In all cases, for the new tree created, $L = F+1$ $\therefore$ the claim follows by structural induction.