CSCI 2200
Foundations of Computer Science (FOCS)

RPI
2016

MIDTERM: 90 Minutes

Last Name: Solutions

First Name: ______________________

RIN: ______________________

Section: ______________________

Answer **ALL** questions. You may use one double sided $8\frac{1}{2} \times 11$ crib sheet.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

1	2	3	4	5	Total
100	40	40	40	40	250

(10 bonus points)

C D
D D
B D
C C
D A

1 Circle one answer per question. 10 points for each correct answer.

(a) Which theorem below is true (n is a natural number, r is a real)?

$\boxed{A}$ n^2 is divisible by 4 if and only if n is divisible by 4.

$\boxed{B}$ If n^2 is divisible by 4 then n is divisible by 4.

$\boxed{C}$ $n^3 + 5$ is odd if and only if n is even.

$\boxed{D}$ If $\sqrt{r}$ is irrational then r is irrational.

C

n even $\to$ n^3 even $\to$ n^5+5 odd

n^3+5 odd $\to$ n^3 even $\to$ n even

(b) How many subsets of $\{a, b, c, d, e, f, g\}$ contain a or g?

$\boxed{A}$ 90

$\boxed{B}$ 92

$\boxed{C}$ 94

$\boxed{D}$ 96

D

2^7 subsets. do not contain a and g = 2^5 Contain a or $g = 2^7 - 2^5$

Inclusion - Exclusion Contain a: 2^6

Contain g: 2^6 $\therefore$ Contain a or $g = 2^6+2^6-2^5 = 2^7-2^5$

Contain $a \wedge g$: 2^5

$2^7-2^5 = 128-32 = 96$

(c) *Estimate* the value of the sum $\sum_{i=0}^{10}\sum_{j=0}^{20} 2^{i+j}$.

$\boxed{A}$ 4×10^6.

$\boxed{B}$ 4×10^9.

$\boxed{C}$ 4×10^{12}.

$\boxed{D}$ 4×10^{15}.

B

$\sum_{i=0}^{10}\sum_{j=0}^{20} 2^{i+j} = \underbrace{\sum_{i=0}^{10} 2^i}_{2^{11}-1} \underbrace{\sum_{j=0}^{20} 2^j}_{2^{21}-1} \approx 2\cdot \underbrace{2^{10}}_{10^3} \cdot 2 \cdot \underbrace{2^{20}}_{10^6} \approx 4\times 10^9$

(d) Which is the correct asymptotic order relationship between 2^{n+1} and 2^{2n}

$\boxed{A}$ $2^{n+1} \in \Theta(2^{2n})$.

$\boxed{B}$ $2^{n+1} \in \omega(2^{2n})$.

$\boxed{C}$ $2^{n+1} \in o(2^{2n})$.

$\boxed{D}$ None of the above.

C

$\frac{2^{n+1}}{2^{2n}} = 2^{-(n-1)} = 2\cdot 2^{-n} \to 0.$

(e) Compute the sum $S = \sum_{i=1}^{3}\sum_{j=1}^{3}(i + j)$

$\boxed{A}$ $S = 30$.

$\boxed{B}$ $S = 32$.

$\boxed{C}$ $S = 34$.

$\boxed{D}$ $S = 36$.

D

$\sum_{i=1}^{n}\sum_{j=1}^{n} i+j = \sum_{i=1}^{n}\sum_{j=1}^{n} i + \sum_{i=1}^{n}\sum_{j=1}^{n} j = \frac{2n \cdot n(n+1)}{2} = n^2(n+1)$

$= n\sum_{i=1}^{n} i + n\sum_{j=1}^{n} j$

$n=3 \therefore 9\cdot 4 = 36.$

(f) What is the last digit of n (i.e. remainder when divided by 10), where $n = 3^{2016} + 4^{2016} + 7^{2016}$?

$3^2 \equiv -1$ $\therefore (3^2)^{1008} \equiv (-1)^{1008} \equiv 1$

$4^2 \equiv 6$ $(4^2)^{1008} \equiv 6^{1008} \equiv 6$ $6^2 \equiv 6 \therefore 6^{1008} \equiv 6$

$7^2 \equiv -1$ $\therefore (7^2)^{1008} \equiv (-1)^{1008} \equiv 1$

$\therefore 3^{2016} + 4^{2016} + 7^{2016} \equiv 1+6+1 \equiv 8.$

D.

$\boxed{A}$ $r = 1$

$\boxed{B}$ $r = 3$

$\boxed{C}$ $r = 6$

$\boxed{D}$ $r = 8$

(g) A friendship network has 5 people. The degrees (number of friends) of the people are 1,1,2,2,3. How many edges (friendship links) are in this friendship network?

Sum of degrees is odd.

D

$\boxed{A}$ 4 edges

$\boxed{B}$ 5 edges

$\boxed{C}$ Not enough information to determine the number of edges

$\boxed{D}$ This friendship network cannot possibly exist

(h) A friendship network has 5 people. The degrees (number of friends) of the people are 0,1,2,3,4. How many edges (friendship links) are in this friendship network?

degree 0 node → max degree is 3.

D

$\boxed{A}$ 4 edges

$\boxed{B}$ 5 edges

$\boxed{C}$ Not enough information to determine the number of edges

$\boxed{D}$ This friendship network cannot possibly exist

(i) Here is some information about ice-skate options:

Colors:	White, Beige, Pink, Yellow, Blue
Sizes:	4,5,6,7,8
Extras:	Tassels, Stripes, Bells

Skates can have any combination of extras (including none). An example skate is (pink; size 5; with stripes and bells). How many types of skates are there?

Skate: C S {extras} $= 5 \times 5 \times 8 = 200.$

5×5 2^3

C

$\boxed{A}$ 150.

$\boxed{B}$ 175.

$\boxed{C}$ 200.

$\boxed{D}$ 225.

(j) In an NBA game, 8 players suit up for a game. At any time only 5 players can be on the floor, playing the game. Using the 8 players, in how many ways can the 5 players on the floor be chosen?

$\binom{8}{5} = \frac{8!}{5!\,3!} = \frac{8 \times 7 \times \cancel{6}}{\cancel{3} \times \cancel{2} \times 1} = 8 \times 7 = 56$

A

$\boxed{A}$ 56.

$\boxed{B}$ 57.

$\boxed{C}$ 58.

$\boxed{D}$ 59.

2 Modular square-root

Let p be prime. **Prove:** $x^2 \equiv y^2 \pmod{p}$ IF AND ONLY IF $x \equiv y \pmod{p}$ or $x \equiv -y \pmod{p}$.
(If your proof uses facts from class or the book, explicitly state them as "from class" or "from the book".)

(1) $x^2 \equiv y^2 \pmod{p} \rightarrow p \mid x^2 - y^2 \rightarrow p \mid (x-y)(x+y)$

$p \mid ab \rightarrow p \mid a$ or $p \mid b$ (from book). Euclid's lemma.

$\therefore$ $p \mid x-y$ or $p \mid x+y$

$\downarrow$

$x \equiv y \pmod{p}$ or $x \equiv -y \pmod{p}$

(2) ~~$p \mid x$ or $p \mid y$~~ $x \equiv y$ or $x \equiv -y \rightarrow x^2 \equiv y^2$

$p \mid x-y \rightarrow p \mid (x-y)(x+y) \rightarrow p \mid x^2 - y^2$
$p \mid x+y \rightarrow p \mid (x-y)(x+y) \rightarrow p \mid x^2 - y^2$

in either case $p \mid x^2 - y^2 \rightarrow x^2 \equiv y^2$

NOTE: to prove an if and only if you MUST prove two implications.

3 Playing with postage

(a) **Prove:** Any postage greater than 7¢ can be made using 3¢ and 5¢ stamps.

Leaping Induction $P(n)$: postage n can be made using 3¢ and 5¢

Base cases
$P(8)$: $3+5$
$P(9)$: $3+3+3$
$P(10)$: $5+5$

Induction $P(n) \rightarrow P(n+3)$
Assume $P(n)$: $n = 3x + 5y \quad x, y \geq 0.$
Prove $P(n+3)$:

[Induction hypothesis]

$$n+3 = 3x + 5y + 3$$
$$= 3(x+1) + 5y$$

That is if you can do n, you can do $n+3$ by just adding 3¢

$\therefore P(n+3)$ is True ∎

(b) **Prove:** For any $k \geq 1$, there is a postage $n \geq k$ that *cannot* be made using 4¢ and 6¢ stamps.

Given k, consider $n = 2k+1 > k$ [prove for a general k].

n cannot be made using 4¢ + 6¢ stamps

Assume n can be made [Proof by contradiction].

$n = 4x + 6y \quad x, y \geq 0.$

that is $\underbrace{2k+1}_{\text{odd}} = \underbrace{4x+6y}_{\text{even}}$ ~~odd~~ FISHY!

Contradiction [odd cannot equal even].

Therefore n cannot be made ∎

4 GCD of consecutive Fibononacci numbers

Let F_n be the nth Fibonacci number: $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$.
Prove: $\gcd(F_n, F_{n+1}) = 1$ for $n \geq 1$. (Any two consecutive Fibonacci numbers are relatively prime.)

Proof By induction $P(n)$: $\gcd(F_n, F_{n+1}) = 1$

Base Case $n=1$ $F_1 = 1$ $F_2 = 1$ $\gcd(1,1) = 1$ ✓.

Strong Induction Assume. $P(1) \wedge P(2) \wedge \ldots P(n)$, Prove $P(n+1)$.

Using Bezout. $\gcd(F_n, F_{n+1}) = 1$ [Bezout]

$\rightarrow x F_n + y F_{n+1} = 1$ (*)

$F_{n+2} = F_{n+1} + F_n \rightarrow x F_{n+2} = x F_{n+1} + x F_n$

$= x F_{n+1} + 1 - y F_{n+1}$ by (*).

$\rightarrow x F_{n+2} + (y - x) F_{n+1} = 1$ a linear combination equals 1 (the min possible)

$\rightarrow \gcd(F_{n+2}, F_{n+1}) = 1$ [Bezout].

Using definition of GCD. and contradiction

suppose $\gcd(F_{n+1}, F_{n+2}) = D > 1$

$F_{n+2} = F_{n+1} + F_n \rightarrow F_n = F_{n+2} - F_{n+1}$ (**)

$D \mid F_{n+2}$ $D \mid F_{n+1}$ since $D = \gcd(F_{n+1}, F_{n+2})$

$\rightarrow$ D divides RHS of (**)

$\rightarrow$ D divides LHS of (**)

$\rightarrow$ D divides F_n

$\rightarrow D \mid F_n$ and $D \mid F_{n+1} \rightarrow \gcd(F_n, F_{n+1}) \geq D > 1$ } Contradiction FISHY!

by IH $\gcd(F_n, F_{n+1}) = 1$

$\therefore \gcd(F_{n+1}, F_{n+2}) = 1$. ∎

5 Recursively defined strings

Define the set of strings $\mathcal{P}$ recursively as follows (The minimality clause is there by default.).

① $0, 1 \in \mathcal{P}$.

② There are two constructor rules: $x \in \mathcal{P} \to x\,0\,x \in \mathcal{P}$;
$x \in \mathcal{P} \to x\,1\,x \in \mathcal{P}$.

Prove that every string in $\mathcal{P}$ is a palindrome (a string that equals its reversal) and has odd length.

Structural Induction $P(s)$: s is a palindrome and $|s|$ is odd.

Base Case. 0 is a Palindrome. and has odd length $P(0)$ is T.
1 is a Palindrome and has odd length $P(1)$ is T.

Induction $x \in P$ and $P(x)$ is True $\therefore x = x^R$ and $|x| = 2k+1$

Constructor rule 1: $(x0x)^R = x^R 0^R x^R = x0x \therefore P(x0x)$ is T
$|x0x| = 2|x|+1$ is odd

Constructor rule 2: $(x1x)^R = x^R 1^R x^R = x1x \therefore P(x1x)$ is T.
$|x1x| = 2|x|+1$ is odd

~~P~~ P is true for all progeny of all constructor rules
$\therefore$ by structural induction, $P(s)$ is true for all $s \in P$.