# MIDTERM: 100 Minutes

Last Name: ____________________

First Name: ____________________

RIN: ____________________

Section: ____________________

Answer **ALL** questions. You may use one double sided $8\frac{1}{2} \times 11$ crib sheet.
**NO COLLABORATION or electronic devices. Any violations result in an F.**
**NO questions** allowed during the test. Interpret and do the best you can.
You **MUST** show your work, **even on multiple choice questions**, to get credit.

## GOOD LUCK!

| 1 | 2 | 3 | 4 | 5 | 6 | Total |
|---|---|---|---|---|---|---|
| | | | | | | |
| 150 | 20 | 20 | 20 | 20 | 20 | 250 |

## 1 Circle one answer per question. 10 points for each correct answer.

(1) All digits of $n > 1$ are 1 (e.g. 111 or 11111). What is the remainder when this number is divided by 4?

$\boxed{\text{A}}$ 0.

$\boxed{\text{B}}$ 1.

$\boxed{\text{C}}$ 2.

$\boxed{\text{D}}$ 3.

$\boxed{\text{E}}$ It depends on the number of digits.

(2) What is the last digit of $n$ (i.e. remainder when divided by 10), where $n = 2017^{2017} - 17^{2017}$?

$\boxed{\text{A}}$ 0

$\boxed{\text{B}}$ 1

$\boxed{\text{C}}$ 3

$\boxed{\text{D}}$ 7

$\boxed{\text{E}}$ 9

(3) **[Hard]** Which theorem below is true ($n$ is a natural number)?

$\boxed{\text{A}}$ $n^2$ is divisible by 10 if and only if $n$ is divisible by 4.

$\boxed{\text{B}}$ $n^2$ is divisible by 8 if and only if $n$ is divisible by 4.

$\boxed{\text{C}}$ $n^2$ is divisible by 4 if and only if $n$ is divisible by 4.

$\boxed{\text{D}}$ $n^2$ is divisible by 2 if and only if $n$ is divisible by 4.

$\boxed{\text{E}}$ None of the above.

(4) Compute the sum $S = \sum_{i=1}^{2} \sum_{j=1}^{2} (i + j)$

$\boxed{\text{A}}$ $S = 6$.

$\boxed{\text{B}}$ $S = 8$.

$\boxed{\text{C}}$ $S = 10$.

$\boxed{\text{D}}$ $S = 12$.

$\boxed{\text{E}}$ $S = 14$.

(5) How many subsets of $\{a, b, c, d, e, f, g\}$ contain $a$ or $g$?

$\boxed{\text{A}}$ 32

$\boxed{\text{B}}$ 64

$\boxed{\text{C}}$ 96

$\boxed{\text{D}}$ 108

$\boxed{\text{E}}$ 128

(6) *Estimate* the value of $2^1 \times 2^2 \times \cdots \times 2^{10} = \prod_{i=1}^{10} 2^i$.

$\boxed{\text{A}}$ $3.6 \times 10^{10}$.

$\boxed{\text{B}}$ $3.6 \times 10^{12}$.

$\boxed{\text{C}}$ $3.6 \times 10^{14}$.

$\boxed{\text{D}}$ $3.6 \times 10^{16}$.

$\boxed{\text{E}}$ $3.6 \times 10^{18}$.

(7) Which is the correct asymptotic order relationship between $f(n) = 2^n$ and $g(n) = \sum_{i=0}^{n} 2^i$

$\boxed{\text{A}}$ $f \in \Theta(g)$.

$\boxed{\text{B}}$ $f \in \omega(g)$.

$\boxed{\text{C}}$ $f \in o(g)$.

$\boxed{\text{D}}$ None of the above.

(8) $\{a_1, a_2, a_3, a_4, a_5\} = \{1, 4, 9, 16, 25\}$ and $\{b_1, b_2, b_3, b_4, b_5\} = \{1, -1, 1, -1, 1\}$. Compute $\sum_{i=1}^{5} \sum_{j=1}^{5} a_i b_j$.

$\boxed{\text{A}}$ 51.

$\boxed{\text{B}}$ 53.

$\boxed{\text{C}}$ 55.

$\boxed{\text{D}}$ 57.

$\boxed{\text{E}}$ 59.

(9) A friendship network (simple graph) has vertices having degree sequence $\boldsymbol{\delta} = [4, 4, 4, 2, 2]$. How many edges (friendship links) are in this friendship network?

$\boxed{\text{A}}$ 7 edges

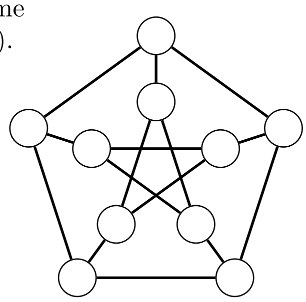
$\boxed{\text{B}}$ 8 edges

$\boxed{\text{C}}$ 9 edges

$\boxed{\text{D}}$ Not enough information to determine the number of edges

$\boxed{\text{E}}$ This friendship network cannot possibly exist

(10) You wish to color the Petersen graph so that linked vertices do not get the same color. What is the minimum number of colors that you need (the chromatic number).



$\boxed{\text{A}}$ 2

$\boxed{\text{B}}$ 3

$\boxed{\text{C}}$ 4

$\boxed{\text{D}}$ 5

$\boxed{\text{E}}$ 6

(11) How many 5-letter strings have no two consecutive letters the same (letters are $a, b, \ldots, z$).

$\boxed{\text{A}}$ $26 \times 25 \times 24 \times 23 \times 22$.

$\boxed{\text{B}}$ $26^5$

$\boxed{\text{C}}$ $\binom{30}{4}$.

$\boxed{\text{D}}$ $26 \times 25^4$.

$\boxed{\text{E}}$ None of the above.

(12) Evaluate $\binom{8}{3} - \binom{8}{5}$

$\boxed{\text{A}}$ 0.

$\boxed{\text{B}}$ 1.

$\boxed{\text{C}}$ 2.

$\boxed{\text{D}}$ 3.

$\boxed{\text{E}}$ 4.

(13) A $10 \times 10$ *binary* array $B$ has $i$ 1's in each row. How many choices are there for $B$?

$\boxed{\text{A}}$ $100^i$.

$\boxed{\text{B}}$ $\binom{100}{10 \times i}$.

$\boxed{\text{C}}$ $\binom{10}{i}^{10}$.

$\boxed{\text{D}}$ $\dfrac{100!}{i!}$.

$\boxed{\text{E}}$ $\dfrac{100!}{(100-i)!}$.

(14) What is the coefficient of $x^3$ in the expansion of $(2-3x)^5$?

$\boxed{\text{A}}$ 360.

$\boxed{\text{B}}$ -360.

$\boxed{\text{C}}$ 720.

$\boxed{\text{D}}$ -720.

$\boxed{\text{E}}$ -1080.

(15) How many integer choices for $z_1, z_2, z_3$ satisfy $0 \leq z_1 \leq z_2 \leq z_3 \leq 10$?

$\boxed{\text{A}}$ $\binom{10}{3} = 120$.

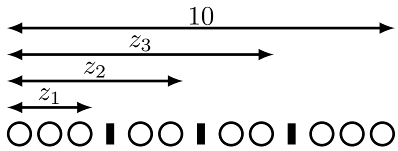
$\boxed{\text{B}}$ $\binom{13}{3} = 286$.

$\boxed{\text{C}}$ $10 \times 9 \times 8 = 720$.

$\boxed{\text{D}}$ $10^3 = 1000$.

$\boxed{\text{E}}$ None of the above.

*Hint: Here is a picture for the case* $z_1 = 3, z_2 = 5, z_3 = 7$.



## 2 Common Divisors Divide the GCD

Prove that every common divisor of $m$ and $n$ divides $\gcd(m, n)$.

**3** **Prove that $\log_{10} 9$ is an irrational number.**

## 4 **Prove by *induction*:** $1+2+\cdots+n=\sum_{i=1}^{n} i \leq n^2$.

(Note: There are many ways to prove this. You are required to prove it by *induction*.)

## 5 A recursively defined set of numbers.

The set $\mathcal{A}$ is recursively defined as shown.
(By default, nothing else is in $\mathcal{A}$ – minimality.)

*Prove* that every number in $\mathcal{A}$ is a multiple of 3.

1. $3 \in \mathcal{A}$.
2. $x, y \in \mathcal{A} \to x + y \in \mathcal{A}$;
   $x, y \in \mathcal{A} \to x - y \in \mathcal{A}$.

## 6 4-friendship-cliques and 3-wars.

Prove: among any 9 people, there are either 4 mutual friends (a 4-clique) or 3 mutual enemies (a 3-war). (You may assume that among any 6 people, either 3 are mutual friends or 3 are mutual enemies.)

**SCRATCH**

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