MIDTERM: 100 Minutes

Last Name: Solutions

First Name: ______

RIN: ______

Section: ______

Answer **ALL** questions. You may use one double sided $8\frac{1}{2} \times 11$ crib sheet.

NO COLLABORATION or electronic devices. Any violations result in an F.

NO questions allowed during the test. Interpret and do the best you can.

You **MUST** show your work, **even on multiple choice questions**, to get credit.

GOOD LUCK!

1	2	3	4	5	6	Total
150	20	20	20	20	20	250

1 Circle one answer per question. 10 points for each correct answer.

(1) Compute the sum $\sum_{n=0}^{10} 2^n$.

$\sum_{i=0}^{n} 2^i = 2^{n+1}-1$

$2^{11}-1 = 2048-1 = 2047.$

- A 1023.
- B 2047.
- C 2048.
- D 4095.
- E None of the above.

(2) What is the correct asymptotic behavior (order analysis) for the sum $S(n) = \sum_{i=1}^{n} i$.

$\frac{1}{2}n(n+1) \in \Theta(n^2)$

- A $S(n) \in O(n)$.
- B $S(n) \in o(n^2)$.
- C $S(n) \in \Theta(n^2)$.
- D $S(n) \in \omega(n^2)$.
- E None of the above.

(3) Which is the correct asymptotic order relationship that describes the sum $S(n) = \sum_{i=0}^{2n} 2^i$

$2^{2n+1}-1 \in \Theta(2^{2n})$

$\frac{2^{2n}}{2^n} = 2^n \to \infty \therefore \omega(2^n)$

- A $S(n) \in \Theta(n^2)$.
- B $S(n) \in \Theta(2^n)$.
- C $S(n) \in \omega(2^n)$.
- D $S(n) \in o(2^n)$.
- E None of the above.

(4) What is the greatest common divisor of 5292 and 6006, gcd(5292, 6006)?

$\gcd(5292, 6006) = \gcd(714, 5292)$
$= \gcd(294, 714)$
$= \gcd(126, 294)$
$= \gcd(42, 126)$
$= \gcd(0, 42)$
$= 42.$

$7 \times 714 = 4900+70+28 = 4998$
$2 \times 294 = 588$
$2 \cdot 126 = 252$

- A 6.
- B 14.
- C 21.
- D 48.
- E None of the above.

(5) What is the last digit of n (the remainder when divided by 10), where $n = 29^9 - 21^7$?

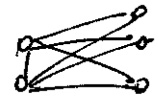
$29 \equiv -1 \pmod{10}$ $\quad 21 \equiv 1$
$\to 29^9 \equiv -1^9 \equiv -1 \pmod{10}$ $\quad \to 21^7 \equiv 1$
$\equiv 9 \pmod{10}$
$29^9 - 21^7 \equiv 9-1 \equiv 8 \pmod{10}.$

- A 0
- B 2
- C 4
- D 6
- E 8

(6) A friendship network (simple graph) has vertices having degree sequence $\delta = [4, 4, 2, 2, 2]$. How many edges (friendship links) are in this friendship network?

$\sum \delta_i = 14 \Rightarrow 2E$

$\rightarrow E = 7$



$\boxed{A}$ 6 edges

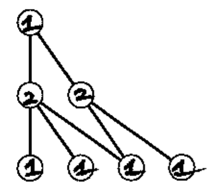
$\boxed{B}$ 7 edges

$\boxed{C}$ 8 edges

$\boxed{D}$ Not enough information to determine the number of edges

$\boxed{E}$ This friendship network cannot possibly exist

(7) You wish to color the graph on the right so that linked vertices do not get the same color. What is the minimum number of colors needed (the chromatic number).



$\boxed{A}$ 2

$\boxed{B}$ 3

$\boxed{C}$ 4

$\boxed{D}$ 5

$\boxed{E}$ 6

(8) Evaluate $\binom{18}{7} - \binom{18}{11}$

$\boxed{A}$ 0

$\boxed{B}$ 1.

$\boxed{C}$ 2.

$\boxed{D}$ 4.

$\boxed{E}$ 36.

$$\binom{n}{k} = \binom{n}{n-k} \rightarrow \binom{18}{7} = \binom{18}{11}$$

$$\rightarrow \binom{18}{7} - \binom{18}{11} = 0$$

(9) How many subsets of $\{a, b, c, d, e, f, g\}$ contain ***at most*** 3 elements.?

$\boxed{A}$ 32.

$\boxed{B}$ 64.

$\boxed{C}$ 128.

$\boxed{D}$ 256.

$\boxed{E}$ None of the above.

$$\binom{7}{0} + \binom{7}{1} + \binom{7}{2} + \binom{7}{3} = \frac{1}{2}\cdot 2^7 = 64.$$

because

$$\underbrace{\binom{7}{0} + \binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} + \binom{7}{7}}_{\text{equal.}} = 2^7$$

(10) *Estimate* the number of possible social networks which could represent the friendship relationships among the 10 friends $A, B, C, D, E, F, G, H, I, J$. (How many different 10-vertex graphs are there?)

$\boxed{A}$ 3.5×10^{12}.

$\boxed{B}$ 3.5×10^{13}.

$\boxed{C}$ 3.5×10^{14}.

$\boxed{D}$ 3.5×10^{15}.

$\boxed{E}$ 3.5×10^{16}.

$2^{10} \approx 10^3$

possible links $= \binom{10}{2} = \frac{10 \times 9}{2} = 45$

each link can be present or not (2 choices)

total choices (Product Rule) $= 2^{45}$

$= (2^{10})^4 \cdot 2^5$

$\approx (10^3)^4 \cdot 2^5$

$= 32 \times 10^{12} \approx 3.2 \times 10^{13}$

(11) In how many ways can you misspell TRIANGLE, assuming you use all the same letters?

[A] 8^8.

[B] $8^8 - 1$.

[C] $8!$.

[D] $8! - 1$.

[E] None of the above.

\# permutations = $8!$

All but one are misspellings

$\to 8! - 1$

(12) In how many ways can you misspell SUCCESS, assuming you use all the same letters?

[A] 379.

[B] 399.

[C] 419.

[D] 449.

[E] None of the above.

\# anagrams $= \frac{7!}{3!2!} = \frac{7\times6\times5\times\cancel{4}^{2}}{\cancel{2}} = 42\times10 = 420$

All but one are misspellings

$\therefore 420 - 1 = 419.$

(13) In how many ways can 10 identical rings be placed on your 5 left-fingers (multiple rings can go on a finger).

[A] 5^{10}.

[B] 10^5.

[C] $\binom{14}{4}$.

[D] $\binom{15}{5}$.

[E] None of the above.

Each ring chooses a finger choose 10 with replacement, order does not matter. Only thing which matters is how many rings on each finger

$n = 5$, $k = 10$ $\quad \binom{n+k-1}{n-1} = \binom{14}{4}$

(14) What is the coefficient of x^3 in the expansion of $(2\sqrt{x} + x)^5$?

[A] 0.

[B] 5.

[C] 40.

[D] 80.

[E] 120

$(2\sqrt{x}+x)^5 = x^{5/2}\underbrace{(2+\sqrt{x})^5}$

need coefficient of $x^{1/2}$.

$\binom{5}{i} 2^i x^{\frac{5-i}{2}} \to \frac{5-i}{2} = \frac{1}{2} \to i = 4.$

$2^4 \cdot \binom{5}{4} = 16 \cdot 5 = \underline{80}$

(15) How many of the numbers in $\{1, 2, \ldots, 1000\}$ are divisible by 10 or 11?

[A] 177.

[B] 179.

[C] 181.

[D] 183.

[E] None of the above.

div by 10: $\left\lfloor \frac{1000}{10} \right\rfloor = 100$

div by 11: $\left\lfloor \frac{1000}{11} \right\rfloor = 90$

div by 11 and 10: $\left\lfloor \frac{1000}{110} \right\rfloor = 9.$

$100 + 90 - 9 = \underline{181.}$

2 Prove that n^2 is divisible by 8 if and only if n is divisible by 4. ($n \in \mathbb{N}$)

Prove two implications

$4 | n \rightarrow 8 | n^2$ (Direct Proof)

Assume $4|n \rightarrow n = 4k. \rightarrow n^2 = 16k^2 = 8\cdot(2k^2)$

$\therefore 8 | n^2$. ∎

$8|n^2 \rightarrow 4|n$. (Contraposition).

Assume 4 does not divide n

So

3 cases
$$\begin{cases} \text{(i) } n = 4k+1 \rightarrow n^2 = 16k^2+8k+1 \rightarrow \mathrm{rem}(n^2, 8) = 1 \\ \text{(ii) } n = 4k+2 \rightarrow n^2 = 16k^2+16k+4 \rightarrow \mathrm{rem}(n^2, 8) = 4 \\ \text{(iii) } n = 4k+3 \rightarrow n^2 = 16k^2+24k+9 \rightarrow \mathrm{rem}(n^2, 8) = 1 \end{cases}$$

In all cases 8 does not divide n^2 ∎

50% understood two proofs and what must be proved in each case, or only proved one correctly.

80% proved one correctly and not other implication.

100% basically proved both implications.

3 **Prove or disprove. There exists $x, y \in \mathbb{Z}$ for which $2x^2 + 5y^2 = 14$.**

Disprove. Proof by contradiction: Suppose $2x^2+5y^2=14$.

Since $2x^2 \geq 0$ it means $5y^2 \leq 14$

so $y^2 \leq \frac{14}{5}$ since y is an integer

$\rightarrow y^2 \leq 2$

since 2 is not a perfect square, the only choices for y are

$y \in \{-1, 0, 1\}$ and $y^2 = 0$ or 1.

$y^2=0 \rightarrow 2x^2 = 14 \rightarrow x^2 = 7 \rightarrow$ 7 is a perfect square, a contradiction

$y^2=1 \rightarrow 2x^2 = 14-5 \rightarrow x^2 = \frac{9}{2} \rightarrow$ x^2 is not an integer which is a contradiction since x is an integer.

Both cases give a contradiction $\rightarrow$ $2x^2+5y^2 \neq 14$

50% showed understanding of problem.
80% used cases and contradiction
100% carried through proof by contradiction or otherwise gave a convincing argument.

4 (Diagonal Binomial Sum) Prove by ***induction***: $\sum_{k=0}^{n} \binom{10+k}{k} = \binom{10+n+1}{n}$ for all integers $n \geq 0$.

$P(n): \sum_{k=0}^{n} \binom{10+k}{k} = \binom{10+n+1}{n}$

Prove $P(n)$ for $n \geq 0$. by induction.

Base Case: $(n=0)$ $P(0)$ claims $\sum_{k=0}^{0} \binom{10+k}{k} = \binom{10+0+1}{0}$

only one term, $k=0$

$\binom{10}{0} = \binom{11}{0}$ ← true because both are 1. ✓

Induction Step: Assume $\sum_{k=0}^{n} \binom{10+k}{k} = \binom{10+n+1}{n}$

Prove $\sum_{k=0}^{n+1} \binom{10+k}{k} = \binom{10+n+2}{n+1}$

$$\sum_{k=0}^{n+1} \binom{10+k}{k} = \sum_{k=0}^{n} \binom{10+k}{k} + \binom{10+n+1}{n+1}$$

$$= \binom{10+n+1}{n} + \binom{10+n+1}{n+1} \quad \text{[Induction Hypothesis].}$$

$$= \binom{10+n+2}{n+1} \checkmark \quad \text{[Pascal's Identity]} \quad \left[\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}\right].$$

$\therefore$ $P(n)$ is true for all $n \geq 0$ by induction ∎

50%: setup correct induction

80%: Proved base case and understood / made some progress in induction step.

100%: Generally correct proof with correct manipulation of binomial coefficients.

Induction step must prove $P(n) \to P(n+1)$.
Not more than 50% if you tried to prove $P(n+1) \to P(n)$

5 How many 10-bit binary strings contain 00 as a substring.

$\#$ 10-bit strings $= 2^{10} = 1024$.

$Q(n) = \#$ of n-bit strings not containing 00

Two cases for these strings.

The start 1 or 01

1 $\underline{(n-1)\text{-bit string not containing } 00}$

01 $\underline{(n-2)\text{-bit string not containing } 00.}$

$Q(n) = Q(n-1) + Q(n-2)$

$Q(1) = \#$ 1-bit strings not containing 00 $= 2$.

$Q(2) = \#$ 2-bit strings not containing 00 $= 3$.

n	1	2	3	4	5	6	7	8	9	10
$Q(n)$	2	3	5	8	13	21	34	55	89	144.

We want $2^{10} - Q(10) = 1024 - 144 = \underline{\underline{880.}}$

50% - showed understanding of problem (eg. tinkering with small n).

80% - Defined $Q(n)$ and approached by build up. Can also develop recursion with # strings containing 00.

100% - Developed correct recursion. OK if addition errors are made.

$T(n) = \#$ of n-bit strings containing 00.

$T(n)$: 3 cases. $\begin{cases} \text{Start } 1 \leftarrow T(n-1) \\ \text{Start } 01 \leftarrow T(n-2) \\ \text{Start } 00 \leftarrow 2^{n-2} \end{cases}$

$T(n) = T(n-1) + T(n-2) + 2^{n-2}$.

$T(1) = 0$

$T(2) = 1$

n	1	2	3	4	5	6	7	8	9	10
$T(n)$	0	1	3	8	19	43	94	201	423	880

6 **Prove or disprove: Any graph has an even number of vertices having odd degree.** (TRUE).
(The graph is undirected and simple, i.e, has no loops or parallel edges.)

Let the even degrees be $\delta_1, \delta_2, \ldots, \delta_m$
Let the odd degrees be $\alpha_1, \alpha_2, \ldots, \alpha_n$
We must prove that n is ~~odd~~ even. Suppose n is odd

$$\text{Sum of degrees} = \underbrace{(\delta_1+\delta_2+\ldots+\delta_m)}_{\text{sum of evens is even}} + \underbrace{(\alpha_1+\alpha_2+\ldots+\alpha_n)}_{\text{sum of } n \text{- odd numbers}}$$

$$= \text{EVEN} + \text{sum of } n \text{ odd numbers}$$

if n is odd then sum of n odd numbers is odd

$$= \text{EVEN} + \text{ODD}$$

$$= \text{ODD}$$ } FISHY. CONTRADICTION

But Sum of degrees $= 2|E| =$ even

$\therefore$ n is ~~odd~~ even $\rightarrow$ # of odd degree vertices is even.

50%: understood problem, showed some tinkering.
80%: Basic idea for proof using handshaking theorem.
100%: Correct proof.

Student may assume sum of odd number of odd numbers is odd, or prove it by induction

$$\alpha_1+\alpha_2+\ldots\alpha_n = \alpha_1 + \ldots \alpha_{n-2} + \underbrace{\alpha_{n-1}\alpha_n}_{\text{sum of 2 odd's is } \cancel{\text{odd}} \text{ even.}}$$

$$\cancel{= \text{ODD}}$$

$$= \text{EVEN} + \underbrace{\alpha_1+\ldots+\alpha_{n-2}}_{\text{apply induction}}$$ hypothesis since $n-2$ is odd

$$= \text{EVEN} + \text{ODD}$$

$$= \text{ODD}.$$