MIDTERM: 90 Minutes

Last Name: Solutions

First Name: ______

RIN: ______

Section: ______

Answer **ALL** questions. You may use one double sided $8\frac{1}{2} \times 11$ crib sheet.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.
You **MUST** show **CORRECT** work, **even on multiple choice questions**, to get credit.

GOOD LUCK!

1	2	3	4	5	6	Total
150	20	20	20	20	20	250

INSTRUCTIONS

1. This is a **closed book** test. No electronics, books, notes, internet, etc.
2. **One** 8.5×11 **double sided crib sheet** is allowed.
3. The test will become available in Submitty at 8am on the test date.
4. Your PDF is due in Submitty by 2pm.
5. By submitting the test you attest that:
   – the work is entirely your own.
   – you obeyed the time limits of the exam.
6. Your submission *must* be typed and submitted as a PDF file.
7. The first page should list your 15 multiple choice answers, like:

(1)	A
(2)	B
(3)	C
(4)	D
⋮	
(15)	A

8. Start each of problems 2–6 on a new page. **SHOW WORK.**
9. After the answers, start a new page to show work for the multiple choice:

(1)	Because x is even
(2)	Because $\sqrt{2}$ is irrational.
(3)	Number of links is $$1 + 2 + \cdots + 10 = 55$$
⋮	
(15)	Because we proved in class that $\ell = n - 1$

   – Some problems may be "easy", so give a one line justification.
   – Some problems may require a detailed reasoning.
   – correct answers: 10 points
   – wrong answers or no work/explanation: 0.
10. **If you don't show correct work, you won't get credit.**
11. Submit with plenty of time to spare. A late test won't be accepted.
    – We won't accept submissions that are even 1 second late.

1 Circle one answer per question. 10 points for each correct answer.

1 nesting
order 1/2

(1) What is the correct asymptotic behavior (order analysis) for the sum $S(n) = \sum_{i=1}^{n} \sqrt{i}$.

$$\theta(n^{(1+1/2)}) = \theta(n^{3/2})$$

E

- [A] $S(n) \in \Theta(n)$.
- [B] $S(n) \in \Theta(n^2)$.
- [C] $S(n) \in \Theta(n^3)$.
- [D] $S(n) \in \Theta(n^4)$.
- [E] None of the above.

(2) What is the correct asymptotic behavior (order analysis) for the sum $S(n) = \sum_{i=1}^{n^2} i$.

Common Sum

$$\theta((n^2)^2) = \theta(n^4)$$

D

- [A] $S(n) \in \Theta(n)$.
- [B] $S(n) \in \Theta(n^2)$.
- [C] $S(n) \in \Theta(n^3)$.
- [D] $S(n) \in \Theta(n^4)$.
- [E] None of the above.

(3) Which is the correct asymptotic order relationship that describes the sum $S(n) = \sum_{i=0}^{n} 2^i$

Common Sum

$$S(n) = 2^{n+1} - 1 \in \theta(2^n).$$

B

- [A] $S(n) \in \Theta(n^2)$.
- [B] $S(n) \in \Theta(2^n)$.
- [C] $S(n) \in \omega(2^n)$.
- [D] $S(n) \in o(2^n)$.
- [E] None of the above.

(4) Estimate $\ln(10^9!)$, that is the logarithm of the factorial of 10^9.

$$\ln(10^9!) = \sum_{i=1}^{10^9} \log i \approx 10^9 \ln 10^9 - 10^9$$

integration

$$\sum_{i=1}^{n} \log i \approx n \ln n - n$$

$$= 9 \cdot 10^9 \ln 10 - 10^9 \approx 10^9 (\underbrace{9 \ln 10 - 1}_{\approx 20}) \approx 2 \times 10^{10}$$

$\ln 10 \gtrsim 2$

A

- [A] 2×10^{10}.
- [B] 2×10^{20}.
- [C] 2×10^{30}.
- [D] 2×10^{40}.
- [E] 2×10^{50}.

these are WAY out.

(5) $\gcd(210, 385) = 210x + 385y$ where $x, y \in \mathbb{Z}$. What are a possible choice for x, y?

$$\gcd(210, 385) = \gcd(175, 210) = \gcd(35, 175) = \gcd(0, 35) = 35$$

$210 \cdot 2 - 385 = 35$ ✓

$x = 2,\ y = -1$

C

- [A] $x = 1, \quad y = 1$.
- [B] $x = -1, \quad y = 1$.
- [C] $x = 2, \quad y = -1$.
- [D] $x = 1, \quad y = 1$.
- [E] None of the above.

(6) What is the remainder when $29^{2019} - 22^{2019}$ is divided by 3?

(mod 3)

- A 0
- B 1
- C 2
- D 3
- E None of the above.

B

$29 \equiv 2$

$(29)^2 \equiv 4 \equiv 1$

$(29)^{2018} \equiv (29^2)^{1009} \equiv 1^{1009} \equiv 1 \therefore 29^{2019} \equiv 29 \equiv 2.$

$22 \equiv 1 \to 22^{2019} \equiv 1 \to 29^{2019} - 22^{2019} \equiv 2 - 1 \equiv 1.$

(7) A friendship network (simple graph) has vertices having degree sequence $\delta = [5, 4, 3, 2, 2]$. How many edges (friendship links) are in this friendship network?

- A 6 edges
- B 7 edges
- C 8 edges
- D Not enough information to determine the number of edges
- E This friendship network cannot possibly exist

E

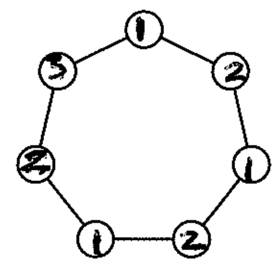
5 vertices, max degree = 4 ∴ can't exist.

(8) You wish to color the graph on the right so that linked vertices do not get the same color. What is the minimum number of colors needed (the chromatic number).

- A 2
- B 3
- C 4
- D 5
- E 6

B

3 colors needed.



(9) At a party with n people, everyone shakes hands with everyone else. How many handshakes occur?

- A $\frac{1}{2}n(n-1)$.
- B $\frac{1}{2}n(n+1)$.
- C $2n$.
- D n^2.
- E None of the above.

A.

Every pair shakes hands

$\binom{n}{2}$ pairs $= \frac{1}{2}n(n-1)$.

$n=2$ 1 hand shake

$n=3$ 3 hand shakes

(10) A connected graph has $n \geq 2$ vertices and no cycles. Does the graph have a degree 1 vertex?

- A Yes, always.
- B No, never.
- C If n is even yes, otherwise no.
- D If n is even no, otherwise yes.
- E None of the above.

A

tree has a leaf

(11) How many subsets of $\{1,2,3,4,5,6,7\}$ contain *one* even number.?

B

A $3+2^4$.

B 3×2^4.

C $\binom{7}{3}$.

D 2^7.

E None of the above.

① Choose even 3 ways

② Choose subset of odd numbers 2^4 ways

Product Rule 3×2^4

(12) There are 4 candy colors. How many goody-bags with 10 candies can you make.

E

A 10^4.

B 4^{10}.

C $\binom{10}{4}$.

D $\binom{14}{4}$.

E None of the above.

$\binom{n+k-1}{k-1} = \binom{13}{3}$

(13) How many sequences of four non-negative integer solutions add up to 10? (e.g. $(4,3,2,1),(3,4,2,1)$).

E

A 10^4.

B 4^{10}.

C $\binom{10}{4}$.

D $\binom{14}{4}$.

E None of the above.

$\underbrace{x_1+x_2+x_3+x_4}_{4 \text{ colors}} = \underset{10 \text{ candies.}}{10}$

$\binom{13}{3}$

Same as Problem (12)

(14) In how many ways can you misspell TEDDY, assuming you use all the same letters?

C

A 57.

B 58.

C 59.

D 60.

E None of the above.

Anagrams $= \frac{5!}{2!} = 60$

Miss spell $\rightarrow$ 59 ways. [60 − 1].

(15) How many of the 1,000 numbers in $\{0,1,\ldots,999\}$ contain the digits 1 or 2?

000, 001, 002, . . . , 999 all 3-digit strings.

C

A 460.

B 474.

C 488.

D 512.

E None of the above.

Do not contain 1 or 2

$\neg(1 \vee 2) = \neg 1 \wedge \neg 2$

Such numbers are 8 digits $\rightarrow 8^3$.

Contain 1 or 2 $= 10^3 - 8^3 = 1000 - 512 = 488$.

$512 = 2^9$.

2 Prove that $\log_2 9$ is not a rational number.

Proof by contradiction

Assume $\log_2 9 = \frac{a}{b}$

} 50%.

$\rightarrow \quad 9 = 2^{a/b}$

Progress = 80%

$\rightarrow \quad 9^b = \left(2^{a/b}\right)^b = 2^a$

odd

even

Full proof 100%.

Contradiction

$\therefore \quad \log_2 9 \neq a/b.$ $\blacksquare$

3 Prove by induction for all $n \geq 1$: $\sum_{i=1}^{n} i2^i = (n-1)2^{n+1} + 2$. P(n).

Proof By Induction

Base Case P(1): $\sum_{i=1}^{1} i2^i = 2^1 = 2$ } Base case is true.

$(1-1)2^{1+1} + 2 = 2$

50%.

Induction Step.

Assume. $\sum_{i=1}^{n} i2^i = (n-1)2^{n+1} + 2$

Prove $\sum_{i=1}^{n+1} i2^i = n2^{n+2} + 2$

$$\sum_{i=1}^{n+1} i2^i = \underbrace{\sum_{i=1}^{n} i2^i}_{(n-1)2^{n+1}+2} + (n+1)2^{n+1}$$

$$= (n-1)2^{n+1} + 2 + (n+1)2^{n+1}$$

$$= (n-1+n+1)2^{n+1} + 2$$

$$= 2n \cdot 2^{n+1} + 2$$

$$= n2^{n+2} + 2$$

Progress gets 80%.

Full proof 100%

Proving P(n+1). By Induction P(n) is true for all $n \geq 1$. $\boxed{\text{///}}$

Tinker 50%.

4 **Find a formula for A_n and prove your answer.** $A_0 = 1$ **and** $A_n = A_{n-1} + n$ **for** $n \geq 1$

$$A_n = \cancel{A_{n-1}} + n$$
$$\cancel{A_{n-1}} = \cancel{A_{n-2}} + n-1$$
$$\cancel{A_{n-2}} = \cancel{A_{n-3}} + n-2$$
$$\vdots$$
$$\cancel{A_2} = \cancel{A_1} + 2$$
$$\cancel{A_1} = \cancel{A_0} + 1$$
$$\cancel{A_0} = 1$$

ADD.

$$A_n = n + n-1 + \cdots + 1 + 1$$
$$= 1 + \underbrace{(1+2+3+\cdots n)}_{\text{common sum}}$$
$$= 1 + \frac{1}{2} n(n+1)$$

80% If got the formula or made good progress

Proof By Induction

Base $A_0 = 1 + \frac{1}{2} \cdot 0 \cdot (0+1) = 1$ ✓

Induction Step

Assume $A_n = 1 + \frac{1}{2} n(n+1)$

Prove $A_{n+1} = 1 + \frac{1}{2}(n+1)(n+2)$

$$A_{n+1} = A_n + n+1$$
$$= 1 + \frac{1}{2} n(n+1) + n+1$$
$$= 1 + \left(\frac{1}{2} n + 1\right)(n+1)$$
$$= 1 + \frac{1}{2}(n+1)(n+2) \quad ✓$$

By induction $A_n = 1 + \frac{1}{2} n(n+1) \quad \forall n \geq 0.$ ∎

100% if correct induction proof

3 Proofs

5 **How many subsets of $\{1, 2, 3, \ldots, 10\}$ have an even sum. (The empty set ϕ has even sum.)**
For example, the subset $\{2, 6\}$ has even sum but $\{2, 7\}$ has odd sum.

TINKER 50%

① 5 even #'s and 5 odd #'s.

5 even #'s: pick any subset $\therefore 2^5$ ways

5 odd #'s: pick an even number

$$\binom{5}{0} + \binom{5}{2} + \binom{5}{4} = \underbrace{1 + 10 + 5}_{16}$$

$$\longrightarrow 2^5 \times 16 = 2^9 = \underline{\underline{512.}}$$

Some Progress 80% → Full result 100%.

② $\text{Sum} = 1+2+3+\cdots+10 = \frac{10 \times 11}{2} = 55$ is odd.

If you pick an even subset, the complement subset is odd.

$\therefore$ every even subset $\longleftrightarrow$ corresponding odd subset

$\longrightarrow$ # even subsets = # odd subsets

$$\underbrace{\#\text{even} + \#\text{odd}}_{2 \times \#\text{even}} = \text{All subsets} = 2^{10}.$$

$$\longrightarrow \#\text{even} = 2^{10}/2 = 2^9 = 512$$

Idea 80% → complete 100%.

③ $E(n)$ = even subsets with $\{1, \ldots, n\}$
$O(n)$ = odd subsets with $\{1, \ldots, n\}$ } Build-Up Counting

You can either chose n or not.

By Symmetry $E(n) = O(n)$

$$E(n) = \begin{cases} 2E(n-1) & n \text{ even} \\ O(n-1) + E(n-1) & n \text{ odd} \end{cases} \qquad E(1) = 1$$

$$O(n) = \begin{cases} 2O(n-1) & n \text{ even} \\ O(n-1) + E(n-1) & n \text{ odd.} \end{cases} \qquad O(1) = 1.$$

80% For Idea

n	1	2	3	4	5	6	7	8	9	10
$E(n)$	1	2	4	8	16	32	64	128	256	512
$O(n)$	1	2	4	8	16	32	64	128	256	512

Full proof 100%.

6 **FOCSbook has n people and each person has $n-2$ friends. Find all n where it is possible?** Precisely state your result and prove it. TINKER!

Tinker 50%.

Claim: This is possible when n is even. Not possible when n is odd.

Proof: Look at # enemies. Each person has 1 enemy.

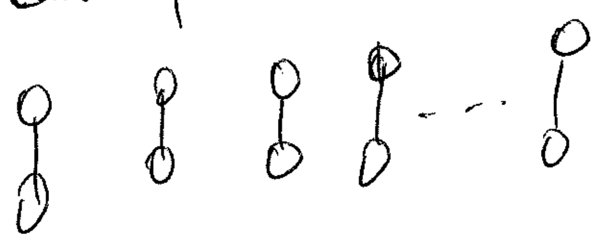
Enemy Network has n-people each with 1 enemy

Idea + some progress → 80%

$\therefore$ if n is odd, degree sum is $n \leftarrow$ (odd)

$\therefore$ not possible by handshaking theorem.

If n is even, make $\frac{n}{2}$ pairs and each pair are enemies:



} Enemy network for even n.

Full Proof 100%.

∎